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ON A SET OF LINEAR EQUATIONS

By E. H. LINFOOT (*Bristol*) and W. M. SHEPHERD (*Bristol*)

[Received 21 April 1938; in revised form 13 July 1938]

In this note we consider the equations

$$\left.\begin{aligned} \frac{\alpha_0}{\lambda}+\frac{\alpha_1}{\lambda+1}+\frac{\alpha_2}{\lambda+2}+\ldots &= 0 \\ \frac{\alpha_0}{\lambda-1}+\frac{\alpha_1}{\lambda}+\frac{\alpha_2}{\lambda+1}+\ldots &= 0 \\ \frac{\alpha_0}{\lambda-2}+\frac{\alpha_1}{\lambda-1}+\frac{\alpha_2}{\lambda}+\ldots &= 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots& \end{aligned}\right\} \tag{1}$$

in the infinitely many unknowns α_0, α_1, α_2,..., where λ is real and not an integer. In §§ 1–3 we prove the following:

THEOREM. *The equations* (1) *have no non-null solution if $\lambda > 0$. If $\lambda < 0$ and p denote the integral part of $|\lambda|$, the equations have just $p+1$ independent solutions, and their most general solution can be written in the form*

$$\alpha_n = c_0\binom{\lambda+n}{n}+c_1\binom{\lambda+n+1}{n}+\ldots+c_p\binom{\lambda+n+p}{n} \quad (n \geqslant 0),$$

where c_0, c_1,..., c_p are arbitrary constants.

In § 4 we give an application to the proof of a result in the theory of Fourier series.

1. Proof of the Theorem

First suppose $\lambda > 0$. We have to show that the only solution of (1) is the 'null' solution $\alpha_0 = \alpha_1 = \ldots = 0$. It is enough to prove this when $0 < \lambda < 1$, since the decreasing of λ by 1 transforms the rth equation into the $(r+1)$th. Suppose therefore that $0 < \lambda < 1$. Equations (1) may be written

$$\sum_{s=0}^{\infty} \frac{\alpha_s}{\lambda-r+s} = 0 \quad (r = 0, 1, 2,\ldots). \tag{1}$$

Subtracting $\lambda-r-1$ times the $(r+2)$th equation from $\lambda-r$ times the $(r+1)$th, we obtain

$$\sum_{s=1}^{\infty} \frac{s\alpha_s}{(\lambda-r+s)(\lambda-r-1+s)} = 0 \quad (r = 0, 1, 2,\ldots). \tag{2}$$

Next, subtracting $\lambda-r-1$ times the $(r+2)$th from $\lambda-r+1$ times the $(r+1)$th of equations (2), we obtain

$$\sum_{s=2}^{\infty} \frac{s(s-1)\alpha_s}{(\lambda-r+s)(\lambda-r-1+s)(\lambda-r-2+s)} = 0 \quad (r = 0, 1, 2,\ldots). \quad (3)$$

Assuming that after $t-1$ operations we have obtained the equations

$$\sum_{s=t-1}^{\infty} \frac{s(s-1)\ldots(s-t+2)\alpha_s}{(\lambda-r+s)(\lambda-r-1+s)\ldots(\lambda-r-t+1+s)} = 0 \quad (r = 0, 1, 2,\ldots), \quad (t)$$

we subtract $\lambda-r-1$ times the $(r+2)$th equation from $\lambda-r+t-1$ times the $(r+1)$th and obtain

$$\sum_{s=t-1}^{\infty} \alpha_s \frac{s(s-1)\ldots(s-t+2)}{(\lambda-r+s)\ldots(\lambda-r-t+s)} \times$$

$$\times\{(\lambda-r+t-1)(\lambda-r-t+s)-(\lambda-r-1)(\lambda-r+s)\} = 0 \quad (r \geqslant 0),$$

which, since

$$(\lambda-r+t-1)(\lambda-r-t+s)-(\lambda-r-1)(\lambda-r+s) = t(s-t+1),$$

can be written

$$\sum_{s=t}^{\infty} \alpha_s \frac{s(s-1)\ldots(s-t+2)(s-t+1)}{(\lambda-r+s)(\lambda-r-1+s)\ldots(\lambda-r-t+s)} = 0 \quad (r \geqslant 0). \quad (t+1)$$

Equations (1) therefore imply equations (t) for $t \geqslant 1$. In particular, they imply the set

$$\left.\begin{array}{r} \dfrac{\alpha_0}{\lambda}+\dfrac{\alpha_1}{\lambda+1}+\dfrac{\alpha_2}{\lambda+2}+\dfrac{\alpha_3}{\lambda+3}+\ldots = 0 \\[2ex] \dfrac{1\,.\,\alpha_1}{\lambda(\lambda+1)}+\dfrac{2\,.\,\alpha_2}{(\lambda+1)(\lambda+2)}+\dfrac{3\,.\,\alpha_3}{(\lambda+2)(\lambda+3)}+\ldots = 0 \\[2ex] \dfrac{1\,.\,2\,.\,\alpha_2}{\lambda(\lambda+1)(\lambda+2)}+\dfrac{2\,.\,3\,.\,\alpha_3}{(\lambda+1)(\lambda+2)(\lambda+3)}+\ldots = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\}. \quad (1)'$$

Conversely, the first r of equations (1) can be obtained from the first r of the equations (1)′ by reversing the above manipulations. Thus (1) and (1)′ are equivalent.

We next eliminate α_1, α_2,... in succession from the first of equations (1)′ by means of the remaining equations. Adding $-\lambda$ times

the second equation to the first, we obtain

$$\frac{\alpha_0}{\lambda}+(1-\lambda)\times$$

$$\times\left\{\frac{\alpha_2}{(\lambda+1)(\lambda+2)}+\frac{2\alpha_3}{(\lambda+2)(\lambda+3)}+\dots+\frac{(s-1)\alpha_s}{(\lambda+s-1)(\lambda+s)}+\dots\right\}=0. \quad (1.1)$$

Adding $\frac{1}{2}(-\lambda)(1-\lambda)$ times the third equation to (1.1), we obtain

$$\frac{\alpha_0}{\lambda}+\frac{(1-\lambda)(2-\lambda)}{1.2}\times$$

$$\times\left\{\frac{1.2.\alpha_3}{(\lambda+1)(\lambda+2)(\lambda+3)}+\dots+\frac{(s-2)(s-1)\alpha_s}{(\lambda+s-2)(\lambda+s-1)(\lambda+s)}+\dots\right\}=0. \quad (1.2)$$

Assume that after $(n-1)$ operations we have obtained

$$\frac{\alpha_0}{\lambda}+\frac{(1-\lambda)(2-\lambda)\dots(n-1-\lambda)}{1.2\dots(n-1)}\left\{\frac{1.2\dots(n-1)\alpha_n}{(\lambda+1)(\lambda+2)\dots(\lambda+n)}+\dots+\right.$$

$$\left.+\frac{(s-n+1)(s-n+2)\dots(s-1)\alpha_s}{(\lambda+s-n+1)(\lambda+s-n+2)\dots(\lambda+s)}+\dots\right\}=0. \quad (1.n-1)$$

Adding $\dfrac{(-\lambda)(1-\lambda)\dots(n-1-\lambda)}{1.2\dots n}$ times the $(n+1)$th equation to $(1.n-1)$, we obtain

$$\frac{\alpha_0}{\lambda}+\frac{(1-\lambda)(2-\lambda)\dots(n-\lambda)}{1.2\dots n}\left\{\frac{1.2\dots n.\alpha_{n+1}}{(\lambda+1)(\lambda+2)\dots(\lambda+n+1)}+\dots+\right.$$

$$\left.+\frac{(s-n)(s-n+1)\dots(s-1)\alpha_s}{(\lambda+s-n)(\lambda+s-n+1)\dots(\lambda+s)}+\dots\right\}=0. \quad (1.n)$$

This equation therefore follows from (1)′ by induction for $n \geqslant 1$. It may be written

$$\alpha_0=-\lambda[n]_\lambda\sum_{s=n+1}^{\infty}t_{n,s}\frac{\alpha_s}{\lambda+s}, \quad (1.n)'$$

where
$$[n]_\lambda=(1-\lambda)\left(1-\frac{\lambda}{2}\right)\dots\left(1-\frac{\lambda}{n}\right)$$

and

$$t_{n,s}=\begin{cases}0 & (s\leqslant n),\\ \dfrac{(s-n)(s-n+1)\dots(s-1)}{(\lambda+s-n)(\lambda+s-n+1)\dots(\lambda+s-1)} & (s\geqslant n+1).\end{cases}$$

Here $0\leqslant t_{n,s}<1$ and, for every n, $t_{n,s}$ is an increasing function of s which tends to 1 as $s\to\infty$.

Now suppose (1) satisfied. Given $\epsilon > 0$, we can choose $n_0 \geqslant 2$ so that

$$|s_{n_0,N}| < \epsilon \text{ for all } N \geqslant n_0, \quad \text{where} \quad s_{n_0,N} = \sum_{n_0 \leqslant s \leqslant N} \frac{\alpha_s}{\lambda+s}.$$

Then

$$\sum_{\nu=n_0}^{N} t_{n,\nu} \frac{\alpha_\nu}{\lambda+\nu} = \sum_{\nu=n_0}^{N} (s_{n_0,\nu} - s_{n_0,\nu-1}) t_{n,\nu}$$

$$= \sum_{\nu=n_0}^{N} s_{n_0,\nu}(t_{n,\nu} - t_{n,\nu+1}) + s_{n_0,N}\, t_{n,N+1},$$

$$\left|\sum_{\nu=n_0}^{N} t_{n,\nu} \frac{\alpha_\nu}{\lambda+\nu}\right| \leqslant \sum_{\nu=n_0}^{N} \epsilon(t_{n,\nu} - t_{n,\nu+1}) + \epsilon t_{n,N+1}$$

$$= \epsilon t_{n,n_0}$$

$$< \epsilon.$$

This holds for $N \geqslant n_0$, $n \geqslant 1$. Hence

$$\left|\sum_{s=n_0}^{\infty} t_{n,s} \frac{\alpha_s}{s+\lambda}\right| \leqslant \epsilon$$

for $n \geqslant 1$. Choose $n = n_0 - 1$; then, by $(1.n)'$,

$$|\alpha_0| < \epsilon.$$

But ϵ is arbitrary; hence $\alpha_0 = 0$.

Since $\alpha_0 = 0$, equations (1) with the first equation removed can now be written

$$\left.\begin{array}{l} \dfrac{\alpha_1}{\lambda} + \dfrac{\alpha_2}{\lambda+1} + \dfrac{\alpha_3}{\lambda+2} + \ldots = 0 \\ \dfrac{\alpha_1}{\lambda-1} + \dfrac{\alpha_2}{\lambda} + \dfrac{\alpha_3}{\lambda+1} + \ldots = 0 \\ \ldots\ldots\ldots\ldots \end{array}\right\}.$$

These equations are identical with the original set (1), except that each α_s is replaced by α_{s+1}. Hence $\alpha_1 = 0$. Repeating this argument, we see that every $\alpha_s = 0$. This proves the first part of the theorem, namely that, if $\lambda > 0$, the equations (1) have no non-null solution.

2. Next suppose $-1 < \lambda < 0$. We have to prove that the only non-null solution of equations (1) is

$$\alpha_n = [n]_{-\lambda}\alpha_0 \quad (n \geqslant 1).$$

We first solve the N equations

$$\left.\begin{aligned} \frac{\alpha_0}{\lambda}+\frac{\alpha_1}{\lambda+1}+\dots+\frac{\alpha_N}{\lambda+N} &= 0 \\ \frac{\alpha_0}{\lambda-1}+\frac{\alpha_1}{\lambda}+\dots+\frac{\alpha_N}{\lambda+N-1} &= 0 \\ \dots\dots\dots\dots &\\ \frac{\alpha_0}{\lambda-N+1}+\frac{\alpha_1}{\lambda-N+2}+\dots+\frac{\alpha_N}{\lambda+1} &= 0 \end{aligned}\right\} \tag{2.1}$$

and then show that on making $N \to \infty$ a solution of (1) is obtained. Now (2.1) will be satisfied by any solution of the equations

$$\left.\begin{aligned} \frac{\alpha_0}{\lambda}+\frac{\alpha_1}{\lambda+1}+\frac{\alpha_2}{\lambda+2}+\dots+\frac{\alpha_N}{\lambda+N} &= 0 \\ \frac{\alpha_1}{\lambda(\lambda+1)}+\frac{2\alpha_2}{(\lambda+1)(\lambda+2)}+\dots+\frac{N\alpha_N}{(\lambda+N-1)(\lambda+N)} &= 0 \\ \dots\dots\dots\dots\dots\dots &\\ \frac{1.2\dots(N-1)\alpha_{N-1}}{\lambda(\lambda+1)\dots(\lambda+N-1)}+\frac{2.3\dots N.\alpha_N}{(\lambda+1)\dots(\lambda+N)} &= 0 \end{aligned}\right\}, \tag{2.2}$$

since (as remarked in the previous manipulation) the first r equations (2.1) can be derived from the first r equations (2.2).

By starting from the last of equations (2.2) and working backwards, their solution is easily conjectured to be

$$\alpha_s = \alpha_{s,N} = \binom{N}{s}\frac{(\lambda+1)(\lambda+2)\dots(\lambda+s)}{(N-\lambda-1)(N-\lambda-2)\dots(N-\lambda-s)}\alpha_0 \quad (s = 1, 2,\dots, N). \tag{2.3}$$

Since the $(r+1)$th of equations (2.2) can be written

$$\sum_{s=r}^{N} \frac{s(s-1)\dots(s-r+1)}{(\lambda+s)(\lambda+s-1)\dots(\lambda+s-r)}\alpha_s = 0,$$

we therefore wish to prove that, for $0 \leqslant r \leqslant N-1$,

$$\sum_{s=r}^{N} \frac{s(s-1)\dots(s-r+1)}{(\lambda+s)(\lambda+s-1)\dots(\lambda+s-r)}\binom{N}{s}\frac{(\lambda+1)\dots(\lambda+s)}{(N-\lambda-1)\dots(N-\lambda-s)} = 0.$$

This can be written

$$\sum_{s=r}^{N} \frac{N(N-1)\dots(N-s+1)}{(N-\lambda-1)\dots(N-\lambda-s)}\cdot\frac{\lambda(\lambda+1)\dots(\lambda+s-r-1)}{1.2\dots(s-r)} = 0$$

(the second factor denoting 1 when $s = r$), or again

$$\sum_{t=0}^{N-r} \frac{N(N-1)...(N-r-t+1)}{(N-\lambda-1)...(N-\lambda-r-t)} \frac{\lambda(\lambda+1)...(\lambda+t-1)}{1.2...t} = 0,$$

on introducing the new variable of summation $t = s-r$. Multiplying through by $[N]_{\lambda+1}$ and writing n for $N-r$, we obtain as the result to be proved

$$\sum_{t=0}^{n} \frac{(-\lambda)(1-\lambda)...(n-t-1-\lambda)}{1.2...(n-t)} . \frac{\lambda(\lambda+1)...(\lambda+t-1)}{1.2...t} = 0 \quad (n = 1, 2, ..., N),$$

an identity which may be verified by considering the coefficient of x^n in the product of the expansions of $(1-x)^\lambda$ and $(1-x)^{-\lambda}$. It has thus been shown that (2.3) is a solution of (2.2) and so of (2.1).

Let $$\alpha_r^* = \lim_{N\to\infty} \alpha_{r,N} = \frac{(\lambda+1)...(\lambda+r)}{1.2...r}\alpha_0 = [r]_{-\lambda}\alpha_0;$$

we have to show that

$$\alpha_r = \alpha_r^* \quad (r = 0, 1, 2, ...)$$

is a solution of (1) or (which is equivalent) of (1)$'$. This follows at once from the fact that, on substituting the values $\alpha_r = \alpha_r^*$, each of the equations (1)$'$ takes the form

$$\frac{\alpha_0}{\lambda}\left\{1-\binom{-\lambda}{1}+\binom{-\lambda}{2}-\binom{-\lambda}{3}+...\right\} = 0,$$

and is satisfied since the (convergent) left-hand side is α_0/λ times the expansion of $(1-x)^{-\lambda}$ with $x = 1$.† Equations (1) are therefore satisfied when $-1 < \lambda < 0$ by

$$\alpha_r = \frac{(\lambda+1)(\lambda+2)...(\lambda+r)}{r!}\alpha_0 \quad (r = 1, 2, ...),$$

that is, by $$\alpha_n = c_1[n]_{-\lambda} \quad (n = 0, 1, 2, ...),$$

where c_1 is any constant. To show that there is no other solution it is enough to prove that two solutions $\alpha_n^{(1)}$ and $\alpha_n^{(2)}$ with $\alpha_0^{(1)} = \alpha_0^{(2)}$ are identical. This follows by substituting the two solutions in (1) and subtracting the corresponding equations; the terms in $\alpha_0^{(1)}-\alpha_0^{(2)}$

† Our thanks are due to the referee for a simplification of the argument at this point.

disappear and we obtain a set of equations

$$\left.\begin{array}{l}\dfrac{\alpha_1^{(1)}-\alpha_1^{(2)}}{\lambda+1}+\dfrac{\alpha_2^{(1)}-\alpha_2^{(2)}}{\lambda+2}+\ldots=0\\[2ex] \dfrac{\alpha_1^{(1)}-\alpha_1^{(2)}}{\lambda}+\dfrac{\alpha_2^{(1)}-\alpha_2^{(2)}}{\lambda+1}+\ldots=0\\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\end{array}\right\}.$$

Since $\lambda+1>0$, it follows by the part of the theorem already proved that

$$\alpha_1^{(1)}-\alpha_1^{(2)},\quad \alpha_2^{(1)}-\alpha_2^{(2)},\quad \ldots$$

are all zero.

3. The conclusions of §§ 1, 2 may be restated as follows:

For $0<\lambda<1$, *the equations*

$$\sum_{n=0}^{\infty}\frac{\alpha_n}{n-r+\lambda}=0\quad(r=0,1,2,\ldots)\tag{0}$$

have as their only solution $\alpha_n=0$ $(n=0,\ 1,\ 2,\ldots)$, *and the equations*

$$\sum_{n=0}^{\infty}\frac{\alpha_n}{n-r+\lambda-1}=0\quad(r=0,1,2,\ldots)\tag{1}$$

have as their most general solution

$$\alpha_n=c_1[n]_{1-\lambda}\quad(n=0,1,2,\ldots),$$

where c_1 *is an arbitrary constant.*

Let $0<\lambda<1$. The equations

$$\sum_{n=0}^{\infty}\frac{\alpha_n}{n-r+\lambda-k-1}=0\quad(r=0,1,2,\ldots)\tag{$k+1$}$$

are identical with the set

$$\sum_{n=0}^{\infty}\frac{\alpha_n}{n-r+\lambda-k}=0\quad(r=1,2,\ldots).$$

It follows that every solution of the equations

$$\sum_{n=0}^{\infty}\frac{\alpha_n}{n-r+\lambda-k}=0\quad(r=0,1,2,\ldots)\tag{k}$$

is also a solution of $(k+1)$. In particular, taking $k=1$,

$$\alpha_n=[n]_{1-\lambda}\quad(n=0,1,2,\ldots)\tag{i}$$

is a solution of the equations

$$\sum_{n=0}^{\infty}\frac{\alpha_n}{n-r+\lambda-2}=0\quad(r=0,1,2,\ldots).\tag{2}$$

A second solution is

$$\left.\begin{array}{l}\alpha_0 = 0 \\ \alpha_n = [n-1]_{1-\lambda} \quad (n = 1, 2, \ldots)\end{array}\right\}. \tag{ii}$$

Therefore $$\alpha_n = \begin{cases} [n]_{1-\lambda}+\mu[n-1]_{1-\lambda} & (n = 1, 2, \ldots), \\ [n]_{1-\lambda} & (n = 0) \end{cases}$$

is always a solution of (2). Thus we can find a solution of (2) in which α_0 and α_1 have any two given values. Moreover, when α_0 and α_1 have assigned values, the solution is unique, since the difference of two such solutions satisfies (0). It follows that (i) and (ii) have the property that every solution of (2) is a linear combination of them. We can replace them by the 'fundamental solutions'

$$\alpha_n = [n]_{1-\lambda} \quad (n = 0, 1, 2, \ldots)$$

and $$\alpha_n = [n]_{2-\lambda} \quad (n = 0, 1, 2, \ldots)$$

on observing that $[n]_{1-\lambda}-[n-1]_{1-\lambda} = [n]_{2-\lambda}$ $(n \geqslant 1)$, while

$$[0]_{1-\lambda} = [0]_{2-\lambda} = 1.$$

In general, equations (k) possess k fundamental solutions with the property that every solution of (k) is a linear combination of these, namely

$$\alpha_n = [n]_{1-\lambda}, \quad \alpha_n = [n]_{2-\lambda}, \quad \ldots, \quad \alpha_n = [n]_{k-\lambda}.$$

For suppose this proved with $k-1$ in place of k. Then

$$\alpha_n = [n]_{k-1-\lambda} \quad (n = 0, 1, 2, \ldots)$$

is one solution of (k), since it satisfies ($k-1$), while

$$\begin{cases} \alpha_0 = 0 \\ \alpha_n = [n-1]_{k-\lambda-1} \quad (n = 1, 2, \ldots) \end{cases}$$

is another. Subtracting, we see that

$$\alpha_n = [n]_{k-\lambda} \quad (n = 0, 1, 2, \ldots)$$

satisfies (k) and

$$\alpha_n = c_{k-1}[n]_{1-\lambda}+c_{k-2}[n]_{2-\lambda}+\ldots+c_0[n]_{k-\lambda} \tag{iii}$$

is a solution of (k) in which the constants c_s can be chosen so as to give $\alpha_0, \alpha_1, \ldots, \alpha_{k-1}$ any assigned values.† Now two solutions $\alpha_n^{(1)}$, $\alpha_n^{(2)}$

† It may be verified without difficulty that the determinant

$$\|[r-1]_{s-\lambda}\| \qquad (r, s = 1, 2, \ldots, k)$$

is equal to $(-1)^{\frac{1}{2}k(k-1)}$ and so does not vanish.

of (k) in which $\alpha_n^{(1)} = \alpha_n^{(2)}$ for $n = 0, 1, \ldots, (k-1)$ are identical, since their difference yields a solution of (0) in the form $\alpha_n = \alpha_{n+k}^{(1)} - \alpha_{n+k}^{(2)}$ $(n \geqslant 0)$ and thus is null. Hence (iii) is the most general solution of (k) and our statement is proved by induction.

Lastly, replace λ by $\lambda+k$, so that the new condition on λ is $-k < \lambda < -k+1$; we obtain: *The most general solution of the equations*

$$\sum_{n=0}^{\infty} \frac{\alpha_n}{n-r+\lambda} = 0 \quad (r = 0, 1, 2, \ldots),$$

where $-k < \lambda < -k+1$, *is*

$$\alpha_n = c_0[n]_{-\lambda} + c_1[n]_{-\lambda-1} + \ldots + c_{k-1}[n]_{-\lambda-k+1},$$

where the c's are arbitrary constants and

$$[n]_{-\lambda-r} = \frac{(\lambda+r+1)(\lambda+r+2)\ldots(\lambda+r+n)}{1.2\ldots n} = \binom{\lambda+r+n}{n}.$$

Since $p = k-1$, this proves the theorem.

4. An application

By means of this theorem, we can obtain a simple proof of the result that the equations

$$\left.\begin{aligned} \sum_{n=0}^{\infty} a_n \cos n\theta &= f(\theta) \quad (|\theta| < \tfrac{1}{2}\pi) \\ \sum_{n=1}^{\infty} a_n \sin n\theta &= g(\theta) \quad (\tfrac{1}{2}\pi < |\theta| < \pi) \end{aligned}\right\}, \tag{1}$$

where $f(\theta)$, $g(\theta)$ are given functions and the series on the left are both Fourier series over $(-\pi, \pi)$, possess at most two independent solutions. It is sufficient to prove that the equations

$$\left.\begin{aligned} \sum_{n=0}^{\infty} a_n \cos n\theta &= 0 \quad (|\theta| < \tfrac{1}{2}\pi) \\ \sum_{n=1}^{\infty} a_n \sin n\theta &= 0 \quad (\tfrac{1}{2}\pi < |\theta| < \pi) \end{aligned}\right\} \tag{2}$$

possess as their most general solution in Fourier series

$$a_{2n} = c(-1)^n[n]_{\frac{1}{2}}; \qquad a_{2n+1} = c(-1)^{n+1}[n]_{\frac{1}{2}} \quad (n = 0, 1, 2, \ldots), \tag{3}$$

where c is an arbitrary constant. That this actually is a solution follows from results already obtained by us.†

† (1), (2). We obtain (3) by taking the difference of the solutions A_n and b_n, given by equations (1) and (5) of (2).

Proof. By a trivial modification of the argument of (**1**), pp. 366–8, it is seen that, if the a_n satisfy (2), then

$$\sum_{n=0}^{\infty} \frac{(-1)^n a_{2n}}{2n-2r-1} = 0 \quad (r = 0, 1, 2,...) \tag{4}$$

and
$$\tfrac{1}{2}\pi a_{2r+1} = -\sum_{n=0}^{\infty} \frac{(-1)^{n+r} a_{2n}}{2n+2r+1} \quad (r = 0, 1, 2,...). \tag{5}$$

Since (4) can be written in the form

$$\sum_{n=0}^{\infty} \frac{\alpha_n}{n-r-\frac{1}{2}} = 0 \quad (r \geqslant 0); \qquad \alpha_n = (-1)^n a_{2n},$$

it follows, by taking $\lambda = -\frac{1}{2}$ in the Theorem, that the most general values of the a_{2n} are

$$a_{2n} = c(-1)^n [n]_{\frac{1}{2}} \quad (n = 0, 1, 2,...).$$

By (5) it then follows that for $r = 0, 1, 2,...$

$$a_{2r+1} = (-1)^{r+1} c \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{[n]_{\frac{1}{2}}}{2n+2r+1}$$
$$= (-1)^{r+1} c [r]_{\frac{1}{2}}$$

by (22) of (**1**). Thus (3) is the most general solution of (2).

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AN ADDITIONAL NOTE ON GENERAL TRANSFORMS OF THE CLASS $L^p(0,\infty)$ $(1 < p \leqslant 2)$

By IDA W. BUSBRIDGE (*Oxford*)

[Received 21 July 1938]

IN a recent paper on this subject,* I assumed in Theorem 2 that $\chi_\lambda(x)/x$ belonged to $L^p(0,\infty)$ for $p \geqslant p_1$ and I showed that, when $\Re(\lambda) > 0$, its χ_λ-transform was

$$\phi(x) = \begin{cases} 0 & (x > 1), \\ x^{\lambda-\frac{1}{2}} & (x < 1). \end{cases}$$

By the theory proved in the first part of Theorem 2, this should belong to $L^{p'}(0,\infty)$ for $2 \leqslant p' \leqslant p_1'$, but this is only true if

$$\Re(\lambda) > \tfrac{1}{2} - 1/p_1'.$$

It follows, therefore, that, *if* $0 < \Re(\lambda) \leqslant \frac{1}{2} - 1/p_1'$, *then there are values of* $p \geqslant p_1$ *for which* $\chi_\lambda(x)/x$ *cannot belong to* $L^p(0,\infty)$.

The following independent proof of this fact is of interest. Let $0 < \Re(\lambda) \leqslant \frac{1}{2} - 1/p_1'$ and suppose that $\chi_\lambda(x)/x$ belongs to $L^P(0,\infty)$, where

$$p_1 \leqslant P \leqslant p_2, \qquad \frac{1}{p_2} = \Re(\lambda) + \tfrac{1}{2}.$$

Then it follows from Lemma 4 of my paper that $\chi_\lambda(x)/x$ belongs to $L^r(0,\infty)$ for all $r \geqslant P$. Now, if

$$\frac{\Omega_\lambda(s)}{\lambda+\frac{1}{2}-s} = \int\limits_0^\infty \frac{\chi_\lambda(x)}{x} x^{s-1}\, dx, \tag{1}$$

we have

$$\left|\frac{\Omega_\lambda(s)}{\lambda+\frac{1}{2}-s}\right| \leqslant \int\limits_0^\infty \left|\frac{\chi_\lambda(x)}{x}\right| x^{\sigma-1}\, dx$$
$$\leqslant \left(\int\limits_0^1 \left|\frac{\chi_\lambda(x)}{x}\right|^r dx\right)^{1/r} \left(\int\limits_0^1 x^{r(\sigma-1)/(r-1)}\, dx\right)^{(r-1)/r} +$$
$$+ \left(\int\limits_1^\infty \left|\frac{\chi_\lambda(x)}{x}\right|^P dx\right)^{1/P} \left(\int\limits_1^\infty x^{P'(\sigma-1)}\, dx\right)^{1/P'}.$$

* *Quart. J. of Math.* (Oxford), 9 (1938), 148–60.

where r may be as large as we please. The right-hand side is uniformly bounded if

$$\frac{1}{r}+\eta \leqslant \sigma \leqslant \frac{1}{P}-\eta \quad (\eta > 0),$$

and it follows that the integral (1) converges uniformly in this strip to an analytic function which is regular for $0 < \sigma < 1/P$. Now $\Omega_\lambda(s)$ satisfies the equation

$$\Omega_\lambda(s)\Omega_\lambda(1-s) = 1 \tag{2}$$

on the line $\sigma = \frac{1}{2}$; it will therefore satisfy it throughout the strip $0 < \sigma < 1/P$, and, when $1/P \leqslant \sigma < 1$, $\Omega_\lambda(s)$ will be defined by equation (2). It follows that $\Omega_\lambda(s)$ can have no zeros in the strip $1/P' < \sigma < 1$ since it can have no poles in the strip $0 < \sigma < 1/P$.

Now the integral on the right-hand side of (1) has been shown to converge on the line $\sigma = 1/p_2 = \Re(\lambda)+\frac{1}{2}$ if $p_2 > P$, and it therefore follows that

$$\Omega_\lambda(\lambda+\tfrac{1}{2}) = 0. \tag{3}$$

But, since $\frac{1}{2} \leqslant 1/p_2 < 1/P$, the line $\sigma = \Re(\lambda)+\frac{1}{2}$ lies within the strip $1/P' < \sigma < 1$ in which $\Omega_\lambda(s)$ has no zeros and we therefore have a contradiction.

If $p_2 = P$ and $\sigma = 1/p_2$, the integral on the right-hand side of (1) converges in mean with exponent P' to $\Omega_\lambda(s)/(\lambda+\frac{1}{2}-s)$ and thus (3) is again true. As before, this gives a contradiction. Hence $\chi_\lambda(x)/x$ cannot belong to $L^P(0,\infty)$ if $p_1 \leqslant P \leqslant p_2$.

THE THEORY OF GENERAL TRANSFORMS FOR FUNCTIONS OF THE CLASS $L^p(0, \infty)$ $(1 < p \leqslant 2)$ (II)

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1. In the first paper on this subject* I considered the analogue, for Watson's general transforms, of Titchmarsh's theorem for Fourier transforms. The conditions of the theorems given there were satisfied by kernels such as $x^{\frac{1}{2}}J_{\nu+1}(x)$, but I showed that there could be no transform theory of the type considered there for either of the kernels

$$\chi_\lambda(x) = \begin{cases} 1 & (x > 1) \\ 0 & (x < 1) \end{cases} \quad (\lambda = \tfrac{1}{2}), \tag{1.1}$$

$$\chi_\lambda(x) = \frac{1}{\pi}\log\left|\frac{1+x}{1-x}\right| \quad (\lambda = \tfrac{1}{2}). \tag{1.2}$$

In this paper I consider the transforms of functions of $L^p(0, \infty)$ for kernels of the latter type.

I shall only consider real kernels and the χ_λ-transforms of real functions, as the analysis is heavy even in this case. I shall adopt the notation and conventions used in my former paper, to which I shall refer as Paper I.

The χ_λ-transform formulae are

$$g(x) = x^{-\lambda+\frac{1}{2}}\frac{d}{dx}\left\{x^{\lambda-\frac{1}{2}}\int_0^\infty \frac{\chi_\lambda(xy)}{y}f(y)\,dy\right\}, \tag{1.3}$$

$$f(x) = x^{-\lambda+\frac{1}{2}}\frac{d}{dx}\left\{x^{\lambda-\frac{1}{2}}\int_0^\infty \frac{\chi_\lambda(xy)}{y}g(y)\,dy\right\}. \tag{1.4}$$

When $\chi_\lambda(x)$ is given by (1.1), these formulae become

$$g(x) = \frac{1}{x}f\left(\frac{1}{x}\right), \qquad f(x) = \frac{1}{x}g\left(\frac{1}{x}\right),$$

and, when $f(x)$ belongs to $L^p(0, \infty)$, we therefore have

$$\int_0^\infty \left|\frac{1}{x}g\left(\frac{1}{x}\right)\right|^p dx = \int_0^\infty |f(x)|^p\,dx. \tag{1.5}$$

* *Quart. J. of Math.* (Oxford), 9 (1938), 148–60.

When $\chi_\lambda(x)$ is given by (1.2), it is well known* that equations (1.3) and (1.4) may formally be written

$$\frac{1}{x}g\left(\frac{1}{x}\right) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(y)}{x-y}\,dy, \qquad f(x) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(1/y)g(1/y)}{x-y}\,dy,$$

$f(x)$ being defined in the interval $(-\infty, 0)$ by means of the equation $f(y) = f(-y)$. Thus $(1/x)g(1/x)$ is the Hilbert transform of $f(x)$, and, if $f(x)$ belongs to $L^p(0,\infty)$, $(1/x)g(1/x)$ also belongs to $L^p(0,\infty)$ and satisfies an inequality of the form†

$$\int_0^\infty \left|\frac{1}{x}g\left(\frac{1}{x}\right)\right|^p dx \leqslant M_p \int_0^\infty |f(x)|^p\,dx. \tag{1.6}$$

It follows from (1.5) and (1.6) that we must seek (in general) for the conditions which must be satisfied by $\chi_\lambda(x)$ if, corresponding to each function $f(x)$ of $L^p(0,\infty)$, the function $g(x)$, defined by (1.3), exists and satisfies an inequality of the form (1.6). This inequality becomes an equality with $M_p = 1$ when $p = 2$, since we then have

$$\int_0^\infty |g(x)|^2\,dx = \int_0^\infty |f(x)|^2\,dx. \tag{1.7}$$

Now it is easy to show formally that, if $F(s)$ and $\Omega_\lambda(s)/(\lambda+\frac{1}{2}-s)$ are the Mellin transforms‡ of $f(x)$ and $\chi_\lambda(x)/x$, then the Mellin transform of $(1/x)g(1/x)$ must be $F(s)\Omega_\lambda(1-s)$. If we are to show, therefore, that $(1/x)g(1/x)$ exists and belongs to $L^p(0,\infty)$ whenever $f(x)$ belongs to $L^p(0,\infty)$, we must impose conditions on $\Omega_\lambda(s)$ which will make $F(1/p+it)\Omega_\lambda(1/p'-it)$ belong to $L^{p'}(-\infty,\infty)$ whenever $F(1/p+it)$ belongs to $L^{p'}(-\infty,\infty)$, and which will make

$$\frac{1}{2\pi i}\int_{1/p-iT}^{1/p+iT} F(s)\Omega_\lambda(1-s)x^{-s}\,ds$$

converge in mean with exponent p as $T \to \infty$.§ Since not every function of $L^{p'}(-\infty,\infty)$ is the Mellin transform of a function of

* See Titchmarsh, *Theory of Fourier Integrals* (Oxford, 1937), p. 230, ex. 1. As I make frequent references to this book, I shall refer to it as *F.I.*

† *F.I.* Theorem 101. ‡ See equation (3.1).

§ The fact that this integral must converge in mean with exponent p follows from a paper by Hille and Tamarkin, *Bull. American Math. Soc.* 39 (1933), 768–74.

$L^p(0,\infty)$, we are faced with the following problem, which has never been completely solved:* What conditions must a function of $L^{p'}(-\infty,\infty)$ satisfy if it is to be the Mellin transform of a function of $L^p(0,\infty)$? The whole theory depends upon the solution of this problem. A new, though by no means final, answer to the question is given in Theorem 3, which is then used in the proof of our principal theorem which we now state.

2. THEOREM 1. *Let $\lambda > 0$ or $< -(\frac{1}{2}-1/P')$, where $1 < P \leqslant 2$ and $1/P+1/P' = 1$. Let $\Omega_\lambda(s)$ be an analytic function of $s = \sigma+it$, which is regular for $1/P' \leqslant \sigma \leqslant 1/P$ and which satisfies the following conditions*:

(i) $\Omega_\lambda(s)\Omega_\lambda(1-s) = 1$ *throughout the strip*;

(ii) $\Omega_\lambda(\sigma+it)$ *and* $\Omega_\lambda(\sigma-it)$ *are conjugate complex numbers*;

(iii) $\Omega_\lambda(\sigma+it)$ *is bounded for* $1/P' \leqslant \sigma \leqslant \frac{1}{2}$;

(iv) $\Omega_\lambda(1/P'+it) = e^{i\xi t}k(t)$, *where ξ is a real constant and $k(t)$ is a complex function of t, whose derivative $k'(t)$ belongs to $L(0,\infty)$.*

Let
$$\frac{\chi_\lambda(x)}{x} = \underset{T\to\infty}{\mathrm{l.i.m.}}\frac{1}{2\pi i}\int\limits_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}\frac{\Omega_\lambda(s)}{\lambda+\frac{1}{2}-s}x^{-s}\,ds. \tag{2.1}$$

Let $P \leqslant p \leqslant 2$ and let $f(x)$ belong to $L^p(0,\infty)$. Then the function $g(x)$, defined by (1.3), *exists and satisfies the inequality*
$$\int\limits_0^\infty\left|\frac{1}{x}g\left(\frac{1}{x}\right)\right|^p dx \leqslant K_p\int\limits_0^\infty |f(x)|^p\,dx. \tag{2.2}$$

If $\lambda > \frac{1}{2}-1/P'$ when $\lambda > 0$ and if $\Omega_\lambda(s)$ also satisfies the condition
(v) $\Omega_\lambda(\sigma+it)/(\lambda+\frac{1}{2}-\sigma-it)$ *belongs to* $L^{1/\sigma}(-\infty,\infty)$ *and*
$$|\Omega_\lambda(\sigma+it)| = o(|t|^2) \quad (|t|\to\infty),$$
uniformly for $\frac{1}{2} \leqslant \sigma \leqslant 1/P$, then the reciprocal formula (1.4) *is also true.*

The proof of this theorem, which covers the exceptional cases mentioned above, is given in § 4. We shall first consider some examples.

Example 1. If $\lambda = \frac{1}{2}$ and $\Omega_\lambda(s) = 1$, then $\chi_\lambda(x)$ is given by (1.1). All the conditions of the theorem are satisfied for any $P > 1$.

* See *F.I.* §§ 4.11 and 4.12, where the corresponding problem for Fourier transforms is discussed.

Example 2. If $\lambda = \frac{1}{2}$ and $\Omega_\lambda(s) = \cot\frac{1}{2}\pi s$, then $\chi_\lambda(x)$ is given by (1.2). Since $\cot\frac{1}{2}\pi s$ is regular for $0 < \sigma < 1$ and $|\cot\frac{1}{2}\pi s| \to 1$ as $|t| \to \infty$ along any line of the strip, conditions (i), (ii), (iii), (v) are clearly satisfied for any $P > 1$. In (iv) take

$$\xi = 0, \qquad k(t) = \Omega_\lambda\left(\frac{1}{P'}+it\right) = \cot\tfrac{1}{2}\pi\left(\frac{1}{P'}+it\right);$$

then
$$|k'(t)| = \pi/\{\cosh \pi t - \cos \pi/P'\},$$

and this is $O(1)$ as $|t| \to 0$ and $O(e^{-\pi|t|})$ as $|t| \to \infty$, so it belongs to $L(0,\infty)$. Thus condition (iv) is also satisfied.

Example 3. Let $\alpha > 0$, $c > 0$ and let

$$\Omega_\lambda(s) = \frac{\alpha+1-s}{\alpha+s} c^{s-\frac{1}{2}}.$$

Then, when $\lambda > 0$,

$$\chi_\lambda(x) = \begin{cases} \dfrac{(2\alpha+1)x^{\alpha+1}}{(\alpha+\frac{1}{2}+\lambda)c^{\alpha+\frac{1}{2}}} & (x < c), \\[2ex] \dfrac{(\alpha+\frac{1}{2}-\lambda)c^{\lambda}}{(\alpha+\frac{1}{2}+\lambda)x^{\lambda-\frac{1}{2}}} & (x > c); \end{cases}$$

when $\lambda < 0$,

$$\chi_\lambda(x) = \begin{cases} \dfrac{(2\alpha+1)x^{\alpha+1}}{(\alpha+\frac{1}{2}+\lambda)c^{\alpha+\frac{1}{2}}} - \dfrac{(\alpha+\frac{1}{2}-\lambda)x^{\frac{1}{2}-\lambda}}{(\alpha+\frac{1}{2}+\lambda)c^{-\lambda}} & (x < c), \\[2ex] 0 & (x > c). \end{cases}$$

$\Omega_\lambda(s)$ is regular for $\sigma > -\alpha$ and conditions (i), (ii), (iii), (v) are clearly satisfied for any $P > 1$. Also

$$\Omega_\lambda\left(\frac{1}{P'}+it\right) = c^{1/P'-\frac{1}{2}}\left\{\frac{2\alpha+1}{\alpha+1/P'+it}-1\right\}e^{it\log c}$$

and this satisfies (iv) with

$$\xi = \log c, \qquad k(t) = c^{1/P'-\frac{1}{2}}\left\{\frac{2\alpha+1}{\alpha+1/P'+it}-1\right\}.$$

In all these examples $\Omega_\lambda(s)$ behaves in a fairly simple manner as $|t| \to \infty$ along any line of the strip considered. When, however, $\Omega_\lambda(s)$ is a combination of gamma functions, as it is for most of the kernels considered in Paper 1, condition (iv) is no longer satisfied. This would not be a great drawback if it were not for the fact that the cosine transform of a function of $L^p(0,\infty)$ has been shown by Hardy and Littlewood* to satisfy an inequality of the form (2.2).

* *Math. Annalen*, 97 (1926), 159–209. See also *F.I.* Theorem 80.

3. Preliminary theorems and lemmas. As the following fundamental theorem for χ_λ-transforms of the class $L^2(0,\infty)$ in which λ and the functions of x are real has never been stated in any published paper, I enunciate it here.

THEOREM 2. *Suppose λ is real and not zero. Then in order that, to each function $f(x)$ of $L^2(0,\infty)$, equation* (1.3) *may define a function $g(x)$ of $L^2(0,\infty)$ and that equations* (1.4) *and* (1.7) *may also be true, it is both necessary and sufficient that $\chi_\lambda(x)$ be defined by equation* (2.1), *where $\Omega_\lambda(s)$ $(s = \frac{1}{2}+it)$ is a complex function of the real variable t, which is integrable over any finite interval and which satisfies the conditions*

(i) $\Omega_\lambda(\frac{1}{2}+it)\Omega_\lambda(\frac{1}{2}-it) = 1$;

(ii) $\Omega_\lambda(\frac{1}{2}+it)$ *and* $\Omega_\lambda(\frac{1}{2}-it)$ *are conjugate complex numbers.*

The extension of this theorem to complex functions and complex values of λ can easily be made.* It will be seen that $\Omega_\lambda(s)$ in Theorem 1 satisfies all the conditions of Theorem 2 on the line $\sigma = \frac{1}{2}$.

The proof of Theorem 1 depends mainly upon the following theorem:

THEOREM 3. *Let $f(x)$ be a real function of the class $L^p(0,\infty)$ and let*

$$F\left(\frac{1}{p}+it\right) = \underset{X\to\infty}{\mathrm{l.i.m.}_{p'}} \int\limits_{X^{-1}}^{X} f(x)x^{1/p+it-1}\,dx \tag{3.1}$$

be its Mellin transform.† *Let $k(t)$ be a complex function of the real variable t which is such that*

(i) *when* $t \geqslant 0$, $k(t) = k(0)+\int\limits_0^t k'(u)\,du$, *where* $k'(t)$ *belongs to* $L(0,\infty)$ *and* $|k(0)| < \infty$;

(ii) $k(t)$ *and* $k(-t)$ *are conjugate complex numbers.*

Let ξ be a real constant. Then

$$h(x,T) = \frac{1}{2\pi}\int\limits_{-T}^{T} F\left(\frac{1}{p}+it\right)k(t)e^{i\xi t}x^{-1/p-it}\,dt \tag{3.2}$$

converges in mean with exponent p, as $T\to\infty$, to a real function $h(x)$ of $L^p(0,\infty)$ such that

$$\int\limits_0^\infty |h(x)|^p\,dx \leqslant K_p \int\limits_0^\infty |f(x)|^p\,dx, \tag{3.3}$$

* Cf. Kober, *Quart. J. of Math.* (Oxford), 8 (1937), 172–85.

† By 'the Mellin transform' of a function of $L^p(0,\infty)$ we shall mean the function defined by equation (3.1).

and the Mellin transform in $L^{p'}$ *of* $h(x)$ *is*

$$F\left(\frac{1}{p}+it\right)k(t)e^{i\xi t}.$$

If we transform this theorem by means of the substitutions

$$x = e^{\eta}, \qquad e^{\eta/p}f(e^{\eta}) = (2\pi)^{-\frac{1}{2}}\phi(\eta), \qquad F\left(\frac{1}{p}+it\right) = \Phi(t),$$

$$e^{\eta/p}h(e^{\eta}, T) = (2\pi)^{-\frac{1}{2}}H(\eta, T), \qquad e^{\eta/p}h(e^{\eta}) = (2\pi)^{-\frac{1}{2}}H(\eta)$$

and then replace η by x, ϕ by f, Φ by F, we obtain the theorem in the following form which is easier to prove.

THEOREM 3′. *Let* $f(x)$ *be a real function of the class* $L^p(-\infty,\infty)$ *and let*

$$F(t) = \underset{a\to\infty}{\mathfrak{L}_{p'}} \frac{1}{\sqrt{(2\pi)}} \int_{-a}^{a} f(x)e^{ixt}\,dx \tag{3.4}$$

be its Fourier transform. Let $k(t)$ *be a complex function of the real variable* t, *which satisfies conditions* (i) *and* (ii) *of Theorem* 3, *and let* ξ *be a real constant. Then*

$$H(x, T) = \frac{1}{\sqrt{(2\pi)}} \int_{-T}^{T} F(t)k(t)e^{-i(x-\xi)t}\,dt \tag{3.5}$$

converges in mean with exponent p, *as* $T\to\infty$, *to a function* $H(x)$ *of* $L^p(-\infty,\infty)$, *such that*

$$\int_{-\infty}^{\infty} |H(x)|^p\,dx \leqslant K_p \int_{-\infty}^{\infty} |f(x)|^p\,dx, \tag{3.6}$$

and the Fourier transform in $L^{p'}(-\infty,\infty)$ *of* $H(x)$ *is*

$$F(t)k(t)e^{i\xi t}.$$

Let $F(t) = F_1(t)+iF_2(t)$, where $F_1(t)$ and $F_2(t)$ are real functions. Then $F_1(t)$ is the cosine transform of $\frac{1}{2}\{f(x)+f(-x)\}$ and $F_2(t)$ is the sine transform of $\frac{1}{2}\{f(x)-f(-x)\}$; $F_1(t)$ is an even function, $F_2(t)$ is an odd function, and each belongs to $L^{p'}(0,\infty)$. Similarly, let $k(t) = k_1(t)+ik_2(t)$. Then, by Theorem 3, condition (ii), $k_1(t)$ is an even function, $k_2(t)$ an odd function, $k_1(0) = k(0)$, $k_2(0) = 0$. Also, by condition (i),

$$|k(t)| \leqslant |k(0)|+\int_0^{\infty} |k'(u)|\,du = K \text{ (say)}.$$

Thus $k(t)$, $k_1(t)$, $k_2(t)$ are all bounded in $(-\infty,\infty)$ and (clearly) $k_1'(t)$ and $k_2'(t)$ both belong to $L(0,\infty)$.

From (3.5) we have

$$H(x,T) = \frac{1}{\sqrt{(2\pi)}} \int_{-T}^{T} [F_1(t)+iF_2(t)][k_1(t)+ik_2(t)]e^{-i(x-\xi)t}\,dt$$

$$= \sqrt{\left(\frac{2}{\pi}\right)} \int_0^T F_1(t)k_1(t)\cos(x-\xi)t\,dt -$$

$$- \sqrt{\left(\frac{2}{\pi}\right)} \int_0^T F_2(t)k_2(t)\cos(x-\xi)t\,dt +$$

$$+ \sqrt{\left(\frac{2}{\pi}\right)} \int_0^T F_1(t)k_2(t)\sin(x-\xi)t\,dt +$$

$$+ \sqrt{\left(\frac{2}{\pi}\right)} \int_0^T F_2(t)k_1(t)\sin(x-\xi)t\,dt. \quad (3.7)$$

Consider the first of these integrals, and write

$$\phi_1(x,t) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^t F_1(u)\cos(x-\xi)u\,du. \quad (3.8)$$

On integrating by parts, we have

$$\sqrt{\left(\frac{2}{\pi}\right)} \int_0^T F_1(t)k_1(t)\cos(x-\xi)t\,dt = k_1(T)\phi_1(x,T) - \int_0^T \phi_1(x,t)k_1'(t)\,dt$$

and thus

$$\int_{-\infty}^{\infty} \left| \sqrt{\left(\frac{2}{\pi}\right)} \int_0^T F_1(t)k_1(t)\cos(x-\xi)t\,dt \right|^p dx$$

$$\leqslant 2^{p-1}|k_1(T)|^p \int_{-\infty}^{\infty} |\phi_1(x,T)|^p\,dx + 2^{p-1} \int_{-\infty}^{\infty} \left| \int_0^T \phi_1(x,t)k_1'(t)\,dt \right|^p dx.$$

We shall show that the right-hand side of this is less than

$$K_p \int_{-\infty}^{\infty} |f(x)|^p\,dx,$$

where K_p depends on p alone. By Hölder's inequality,

$$\left| \int_0^T \phi_1(x,t)k_1'(t)\,dt \right|^p \leqslant \int_0^T |\phi_1(x,t)|^p |k_1'(t)|\,dt \left(\int_0^T |k_1'(t)|\,dt \right)^{p/p'}.$$

Hence

$$\int_{-\infty}^{\infty}\left|\int_0^T \phi_1(x,t)k_1'(t)\,dt\right|^p dx \leqslant \left(\int_0^T |k_1'(t)|\,dt\right)^{p/p'} \int_{-\infty}^{\infty} dx \int_0^T |\phi_1(x,t)|^p |k_1'(t)|\,dt$$

$$\leqslant \left(\int_0^\infty |k_1'(t)|\,dt\right)^{p/p'} \int_0^\infty |k_1'(t)|\,dt \int_{-\infty}^{\infty} |\phi_1(x,t)|^p\,dx,$$

and thus we have

$$\int_{-\infty}^{\infty}\left|\sqrt{\left(\frac{2}{\pi}\right)}\int_0^T F_1(t)k_1(t)\cos(x-\xi)t\,dt\right|^p dx$$

$$\leqslant 2^{p-1}|k_1(T)|^p \int_{-\infty}^{\infty} |\phi_1(x,T)|^p\,dx +$$

$$+2^{p-1}\left(\int_0^\infty |k_1'(t)|\,dt\right)^{p/p'} \int_0^\infty |k_1'(t)|\,dt \int_{-\infty}^{\infty} |\phi_1(x,t)|^p\,dx. \quad (3.9)$$

Now consider $\phi_1(x,t)$. Since $\cos(x-\xi)u$ belongs to $L^p(0,t)$, it follows from (3.4) that

$$\phi_1(x,t) = \lim_{a\to\infty} \frac{2}{\pi}\int_0^t \cos(x-\xi)u\,du \int_0^a \frac{f(y)+f(-y)}{2}\cos uy\,dy$$

$$= \frac{2}{\pi}\int_0^\infty \frac{f(y)+f(-y)}{2}\,dy \int_0^t \cos(x-\xi)u\cos uy\,du$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(y)+f(-y)}{2}\,\frac{\sin(x-\xi-y)t}{x-\xi-y}\,dy$$

$$= \frac{1}{\pi}\sin(x-\xi)t\int_{-\infty}^{\infty} \frac{\frac{1}{2}\{f(y)+f(-y)\}\cos yt}{(x-\xi)-y}\,dy -$$

$$-\frac{1}{\pi}\cos(x-\xi)t\int_{-\infty}^{\infty} \frac{\frac{1}{2}\{f(y)+f(-y)\}\sin yt}{(x-\xi)-y}\,dy. \quad (3.10)$$

Now, if $$g(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\frac{1}{2}\{f(y)+f(-y)\}\cos yt}{x-y}\,dy,$$

it follows from the theory of Hilbert transforms* that

$$\int_{-\infty}^{\infty} |g(x-\xi)|^p\,dx = \int_{-\infty}^{\infty} |g(x)|^p\,dx$$

$$\leqslant K_p \int_{-\infty}^{\infty} |\tfrac{1}{2}\{f(y)+f(-y)\}\cos yt|^p\,dy$$

$$\leqslant K_p \int_{-\infty}^{\infty} |f(y)|^p\,dy$$

and the second integral in (3.10) satisfies a similar inequality. Hence we have

$$\int_{-\infty}^{\infty} |\phi_1(x,t)|^p\,dx \leqslant K_p \int_{-\infty}^{\infty} |f(y)|^p\,dy$$

for all values of t. Since $k_1(t)$ is bounded in $(0,\infty)$ and $k_1'(t)$ belongs to $L(0,\infty)$, it now follows from (3.9) that

$$\int_{-\infty}^{\infty} \left|\sqrt{\left(\frac{2}{\pi}\right)} \int_0^T F_1(t)k_1(t)\cos(x-\xi)t\,dt\right|^p dx \leqslant K_p \int_{-\infty}^{\infty} |f(y)|^p\,dy.$$

All the other integrals in (3.7) can be shown to satisfy a similar inequality, and we therefore have, by Minkowski's inequality,

$$\int_{-\infty}^{\infty} |H(x,T)|^p\,dx \leqslant K_p \int_{-\infty}^{\infty} |f(x)|^p\,dx. \tag{3.11}$$

The theorem now follows from the following lemma which is due to Offord:†

Lemma. *If $g(t)$ belong to $L^q(-\infty,\infty)$ $(1<q<\infty)$, and if*

$$f(x,a) = \frac{1}{\sqrt{(2\pi)}} \int_{-a}^{a} g(t)e^{-ixt}\,dt$$

satisfy the inequality

$$\int_{-\infty}^{\infty} |f(x,a)|^r\,dx \leqslant M \quad (1<r<\infty)$$

for all a, then $f(x,a)$ converges in mean with exponent r as $a\to\infty$ to a function $f(x)$ of $L^r(-\infty,\infty)$, and $g(t)$ is the Fourier transform in $L^q(-\infty,\infty)$ of $f(x)$.

* *F.I.* Theorem 101.

† A. C. Offord, *Bull. American Math. Soc.* 41 (1935), 427–36.

Since, in Theorem 3′, $F(t)k(t)e^{i\xi t}$ belongs to $L^{p'}(-\infty,\infty)$ and $H(x,T)$ satisfies (3.11), it follows from the lemma that $H(x,T)$ converges in mean with exponent p to a function $H(x)$ of $L^p(-\infty,\infty)$ whose Fourier transform in $L^{p'}(-\infty,\infty)$ is $F(t)k(t)e^{i\xi t}$; letting $T\to\infty$ in (3.11), we have (3.6).

4. The proof of Theorem 1. This falls into three parts:

(1) the proof of the existence of the χ_λ-transform of every function of the class $L^P(0,\infty)$;

(2) the proof of the existence of the χ_λ-transform of every function of the class $L^p(0,\infty)$ for every p such that $P\leqslant p\leqslant 2$;

(3) the proof of the reciprocal formula (1.4).

We must show first, however, that $\chi_\lambda(x)/x$ belongs to $L^{p'}(0,\infty)$ for $2\leqslant p'\leqslant P'$, this being a necessary and sufficient condition for the existence of the integral on the right-hand side of (1.3) as an L-integral for all functions considered. It follows from (2.1), by the properties of limits-in-mean, that

$$\frac{\chi_\lambda(x)}{x}=\frac{d}{dx}\left\{\frac{1}{2\pi i}\int\limits_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{\Omega_\lambda(s)}{\lambda+\frac{1}{2}-s}\,\frac{x^{1-s}}{1-s}\,ds\right\}.$$

Since $\lambda>0$ or $<-(\frac{1}{2}-1/P')$, the integrand is regular for $1/P'\leqslant\sigma\leqslant\frac{1}{2}$ and it tends uniformly to zero as $|t|\to\infty$, by (iii). We may therefore move the line of integration to $\sigma=1/p'$, giving

$$\frac{\chi_\lambda(x)}{x}=\frac{d}{dx}\left\{\frac{1}{2\pi i}\int\limits_{1/p'-i\infty}^{1/p'+i\infty}\frac{\Omega_\lambda(s)}{\lambda+\frac{1}{2}-s}\,\frac{x^{1-s}}{1-s}\,ds\right\}.$$

Since $\Omega_\lambda(s)/(\lambda+\frac{1}{2}-s)$ belongs to $L^p(-\infty,\infty)$ when $s=1/p'+it$, it follows from the theory of Mellin transforms* that $\chi_\lambda(x)/x$ belongs to $L^{p'}(0,\infty)$ and that

$$\frac{\chi_\lambda(x)}{x}=\underset{T\to\infty}{\mathfrak{L}_{p'}}\,\frac{1}{2\pi i}\int\limits_{1/p'-iT}^{1/p'+iT}\frac{\Omega_\lambda(s)}{\lambda+\frac{1}{2}-s}\,x^{-s}\,ds. \tag{4.1}$$

(1) Let $f(x)$ belong to $L^P(0,\infty)$ and let $F(1/P+it)$ be its Mellin transform defined by (3.1). Then, by Parseval's theorem for Mellin transforms of the class L^P,† we have

$$\int\limits_0^\infty\frac{\chi_\lambda(xy)}{y}f(y)\,dy=\frac{1}{2\pi i}\int\limits_{1/P-i\infty}^{1/P+i\infty}F(s)\frac{\Omega_\lambda(1-s)}{\lambda-\frac{1}{2}+s}x^s\,ds. \tag{4.2}$$

* *F.I.* Theorem 86. † *F.I.* Theorem 87.

Now it follows from conditions (ii) and (iv) that, when $s = 1/P+it$, $\Omega_\lambda(1-s)$ satisfies the conditions imposed on $k(t)e^{i\xi t}$ in Theorem 3; hence

$$h(x) = \underset{T\to\infty}{\mathfrak{L}_P} \frac{1}{2\pi i} \int\limits_{1/P-iT}^{1/P+iT} F(s)\Omega_\lambda(1-s)x^{-s}\, ds \tag{4.3}$$

exists and

$$\int\limits_0^\infty |h(x)|^P\, dx \leqslant K_P \int\limits_0^\infty |f(x)|^P\, dx. \tag{4.4}$$

Now define $g(x)$ by means of the equation

$$g(x) = \frac{1}{x}h\left(\frac{1}{x}\right); \tag{4.5}$$

then we must show that this satisfies equation (1.3) almost everywhere.

When $\lambda > 0$, $x^{-\lambda-\frac{1}{2}}$ belongs to $L^{P'}(y,\infty)$ for $y > 0$, and it follows from (4.3), by the properties of limits-in-mean, that

$$\int\limits_y^\infty x^{-\lambda-\frac{1}{2}}h(x)\, dx = \frac{1}{2\pi i} \lim_{T\to\infty} \int\limits_y^\infty x^{-\lambda-\frac{1}{2}}\, dx \int\limits_{1/P-iT}^{1/P+iT} F(s)\Omega_\lambda(1-s)x^{-s}\, ds$$

$$= \frac{1}{2\pi i} \lim_{T\to\infty} \int\limits_{1/P-iT}^{1/P+iT} F(s)\Omega_\lambda(1-s)\, ds \int\limits_y^\infty x^{-\lambda-s-\frac{1}{2}}\, dx$$

$$= \frac{1}{2\pi i} \int\limits_{1/P-i\infty}^{1/P+i\infty} \frac{F(s)\Omega_\lambda(1-s)}{\lambda-\frac{1}{2}+s} y^{-\lambda+\frac{1}{2}-s}\, ds,$$

the inversion of the order of integration being justified by the uniform convergence of the x-integral with respect to t. Putting $x = u^{-1}$, we have by (4.5)

$$\int\limits_0^{1/y} u^{\lambda-\frac{1}{2}}g(u)\, du = \frac{1}{2\pi i} \int\limits_{1/P-i\infty}^{1/P+i\infty} \frac{F(s)\Omega_\lambda(1-s)}{\lambda-\frac{1}{2}+s} y^{-\lambda+\frac{1}{2}-s}\, ds$$

and thus

$$\int\limits_0^x u^{\lambda-\frac{1}{2}}g(u)\, du = \frac{x^{\lambda-\frac{1}{2}}}{2\pi i} \int\limits_{1/P-i\infty}^{1/P+i\infty} \frac{F(s)\Omega_\lambda(1-s)}{\lambda-\frac{1}{2}+s} x^s\, ds$$

$$= x^{\lambda-\frac{1}{2}} \int\limits_0^\infty \frac{\chi_\lambda(xy)}{y} f(y)\, dy$$

by (4.2). Hence $g(x)$ satisfies (1.3) in this case.

When $\lambda < -(\frac{1}{2}-1/P')$, $x^{-\lambda-\frac{1}{2}}$ belongs to $L^{P'}(0,y)$, and in this case we find that

$$-\int_0^y x^{-\lambda-\frac{1}{2}}h(x)\,dx = \frac{1}{2\pi i}\int_{1/P-i\infty}^{1/P+i\infty} \frac{F(s)\Omega_\lambda(1-s)}{\lambda-\frac{1}{2}+s}y^{-\lambda+\frac{1}{2}-s}\,ds$$

and thus (as above)

$$-\int_x^\infty u^{\lambda-\frac{1}{2}}g(u)\,du = x^{\lambda-\frac{1}{2}}\int_0^\infty \frac{\chi_\lambda(xy)}{y}f(y)\,dy,$$

from which equation (1.3) again follows. Finally, from (4.4) and (4.5), we have the inequality (2.2) in the case $p = P$.

(2) When $p = 2$, by Theorem 2, $g(x)$ exists, $(1/x)g(1/x)$ belongs to $L^2(0,\infty)$, and

$$\int_0^\infty \left|\frac{1}{x}g\left(\frac{1}{x}\right)\right|^2 dx = \int_0^\infty |f(x)|^2\,dx.$$

Hence the equation

$$\frac{1}{x}g\left(\frac{1}{x}\right) = -x^{\lambda+\frac{1}{2}}\frac{d}{dx}\left\{x^{-\lambda+\frac{1}{2}}\int_0^\infty \chi_\lambda\left(\frac{y}{x}\right)\frac{f(y)}{y}\,dy\right\},$$

which is derived from (1.3) by writing $1/x$ for x, defines a linear functional transformation which transforms the class $L^P(0,\infty)$ into itself and the class $L^2(0,\infty)$ into itself. It therefore follows from Lemmas 1 and 2 of Paper 1 that the transformation can be extended to any exponent p such that $P \leqslant p \leqslant 2$ and that, if $f(x)$ belongs to $L^p(0,\infty)$, then

$$\int_0^\infty \left|\frac{1}{x}g\left(\frac{1}{x}\right)\right|^p dx \leqslant K_P^{\frac{P(2-p)}{p(2-P)}}\int_0^\infty |f(x)|^p\,dx = K_p\int_0^\infty |f(x)|^p\,dx.$$

This completes the proof of the first part of the theorem.

(3) So far no use has been made of condition (v); we now suppose that this is satisfied. Consider the function

$$\phi(x) = \underset{T\to\infty}{\mathfrak{L}_{p'}}\frac{1}{2\pi i}\int_{1/p'-iT}^{1/p'+iT}\frac{\Omega_\lambda(1-s)}{\lambda-\frac{1}{2}+s}x^{-s}\,ds. \tag{4.6}$$

Since the integrand belongs to $L^p(-\infty,\infty)$ on the line $\sigma = 1/p'$, $\phi(x)$ exists and belongs to $L^{p'}(0,\infty)$. Since $\lambda > \frac{1}{2}-1/P'$ or $< -(\frac{1}{2}-1/P')$,

the integrand is regular in the strip $1/P' \leqslant \sigma \leqslant \frac{1}{2}$ and it is $o(|t|)$ as $|t| \to \infty$. It follows that, just as (4.1) was deduced from (2.1), so by integrating with respect to x first and then moving the line of integration to $\sigma = \frac{1}{2}$, we can show that

$$\phi(x) = \underset{T\to\infty}{\text{l.i.m.}} \frac{1}{2\pi i} \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \frac{\Omega_\lambda(1-s)}{\lambda-\frac{1}{2}+s} x^{-s}\, ds.$$

This equation may be written

$$\lim_{T\to\infty} \int_0^\infty \left| \phi(x) - \frac{1}{2\pi} \int_{-T}^{T} \frac{\Omega_\lambda(\frac{1}{2}-it)}{\lambda+it} x^{-\frac{1}{2}-it}\, dt \right|^2 dx = 0$$

and thus, on writing $-t$ for t and $1/y$ for x, we have

$$\lim_{T\to\infty} \int_0^\infty \left| \frac{1}{y}\phi\left(\frac{1}{y}\right) - \frac{1}{2\pi} \int_{-T}^{T} \frac{\Omega_\lambda(\frac{1}{2}+it)}{\lambda-it} y^{-\frac{1}{2}-it}\, dt \right|^2 dy = 0.$$

Hence, by (2.1), $$\frac{1}{y}\phi\left(\frac{1}{y}\right) = \frac{\chi_\lambda(y)}{y},$$

and thus $$\phi(x) = \chi_\lambda\left(\frac{1}{x}\right). \tag{4.7}$$

But $\phi(x)$ belongs to $L^{p'}(0,\infty)$, and we have proved that $(1/x)g(1/x)$ belongs to $L^p(0,\infty)$; hence

$$\int_0^\infty \phi\left(\frac{y}{x}\right)\frac{1}{y}g\left(\frac{1}{y}\right) dy = \int_0^\infty \chi_\lambda\left(\frac{x}{y}\right)\frac{1}{y}g\left(\frac{1}{y}\right) dy \tag{4.8}$$

exists.

Now, if $p = P$, it follows from Theorem 3 and equations (4.3) and (4.5) that $F(s)\Omega_\lambda(1-s)$ $(s = 1/P+it)$ is the Mellin transform in $L^{P'}(-\infty,\infty)$ of $(1/x)g(1/x)$, but it does not follow immediately that this is true when p differs from P. Equation (4.2), however, holds when P is replaced by p, and we therefore have

$$\frac{1}{x}g\left(\frac{1}{x}\right) = -x^{\lambda+\frac{1}{2}}\frac{d}{dx}\left\{x^{-\lambda+\frac{1}{2}} \int_0^\infty \chi_\lambda\left(\frac{y}{x}\right)\frac{f(y)}{y}\, dy\right\}$$

$$= -x^{\lambda+\frac{1}{2}}\frac{d}{dx}\left\{\frac{1}{2\pi i} \int_{1/p-i\infty}^{1/p+i\infty} \frac{F(s)\Omega_\lambda(1-s)}{\lambda-\frac{1}{2}+s} x^{-\lambda+\frac{1}{2}-s}\, ds\right\}. \tag{4.9}$$

But, if $G(1/p+it)$ is the Mellin transform of $(1/x)g(1/x)$, then*

$$\frac{1}{x}g\left(\frac{1}{x}\right) = \underset{T\to\infty}{\mathfrak{L}_p}\, \frac{1}{2\pi i} \int\limits_{1/p-iT}^{1/p+iT} G(s)x^{-s}\, ds,$$

and, since $x^{-\lambda-\frac{1}{2}}$ belongs to $L^{p'}(x,\infty)$ $(\lambda > 0)$ and to $L^{p'}(0,x)$ $(\lambda < -\frac{1}{2}+1/p')$, it follows from this in the usual way that

$$\frac{1}{x}g\left(\frac{1}{x}\right) = -x^{\lambda+\frac{1}{2}}\frac{d}{dx}\left\{\frac{1}{2\pi i}\int\limits_{1/p-i\infty}^{1/p+i\infty} \frac{G(s)}{\lambda-\frac{1}{2}+s}x^{-\lambda+\frac{1}{2}-s}\, ds\right\}. \tag{4.10}$$

The equation

$$G(s) = F(s)\Omega_\lambda(1-s) \quad (s = 1/p+it)$$

now follows easily from (4.9) and (4.10) by the uniqueness theorem for Mellin transforms of the class L.†

We can now apply Parseval's theorem for Mellin transforms of the class L^p to (4.8) and we have, by the last result and (4.6),

$$\int\limits_0^\infty \chi_\lambda\left(\frac{x}{y}\right)\frac{1}{y}g\left(\frac{1}{y}\right) dy = \frac{1}{2\pi i}\int\limits_{1/p-i\infty}^{1/p+i\infty} F(s)\Omega_\lambda(1-s)\frac{\Omega_\lambda(s)}{\lambda+\frac{1}{2}-s}x^{1-s}\, ds$$

$$= \frac{1}{2\pi i}\int\limits_{1/p-i\infty}^{1/p+i\infty} F(s)\frac{x^{1-s}}{\lambda+\frac{1}{2}-s}\, ds$$

by condition (i). Hence, putting $y = 1/u$, we have

$$x^{\lambda-\frac{1}{2}}\int\limits_0^\infty \frac{\chi_\lambda(xu)}{u}g(u)\, du = \frac{1}{2\pi i}\int\limits_{1/p-i\infty}^{1/p+i\infty} F(s)\frac{x^{\lambda+\frac{1}{2}-s}}{\lambda+\frac{1}{2}-s}\, ds. \tag{4.11}$$

Now
$$f(u) = \underset{T\to\infty}{\mathfrak{L}_p}\, \frac{1}{2\pi i}\int\limits_{1/p-iT}^{1/p+iT} F(s)u^{-s}\, ds,$$

and, since $u^{\lambda-\frac{1}{2}}$ belongs to $L^{p'}(0,x)$ $(\lambda > \frac{1}{2}-1/p')$ and to $L^{p'}(x,\infty)$ $(\lambda < 0)$, we have

$$x^{\lambda-\frac{1}{2}}f(x) = \frac{d}{dx}\left\{\frac{1}{2\pi i}\int\limits_{1/p-i\infty}^{1/p+i\infty} F(s)\frac{x^{\lambda+\frac{1}{2}-s}}{\lambda+\frac{1}{2}-s}\, ds\right\}$$

* The fact that this equation can be written as a limit-in-mean with exponent p follows from the work of Hille and Tamarkin, to which we have already referred.

† See *F.I.* § 6.1.

in both cases, and thus by (4.11)

$$f(x) = x^{-\lambda+\frac{1}{2}} \frac{d}{dx}\left\{x^{\lambda-\frac{1}{2}} \int_0^\infty \frac{\chi_\lambda(xu)}{u} g(u)\, du\right\}.$$

This completes the proof.

5. We conclude with a few remarks on the subject of self-reciprocal functions of the class $L^p(0,\infty)$. For kernels of the type considered in Paper 1, theorems completely analogous to those of Hardy and Titchmarsh for Fourier transforms* are true. Such a function must belong to $L^r(0,\infty)$ for $p \leqslant r \leqslant p'$ and therefore to $L^2(0,\infty)$.†

If $f(x)$ is self-reciprocal for a kernel of the type considered in this paper, however, both $f(x)$ and $(1/x)f(1/x)$ must belong to $L^p(0,\infty)$. It easily follows that $x^{\sigma-1/p}f(x)$ must belong to $L^p(0,\infty)$ for $1/p' \leqslant \sigma \leqslant 1/p$, but that $f(x)$ need not belong to $L^2(0,\infty)$. The following theorem, which we state without proof, gives the necessary and sufficient conditions for a function $f(x)$ to be self-reciprocal. Owing to the asymmetry of the theory of Mellin transforms about the number 2, it is impossible to place all the conditions on $F(s)$.

THEOREM 4. *Let $\Omega_\lambda(s)$ satisfy conditions* (i) *to* (iv) *of Theorem* 1 *and let $\chi_\lambda(y)$ be defined by equation* (2.1). *Then the necessary and sufficient conditions for $f(x)$ of $L^p(0,\infty)$ $(p \geqslant P)$ to be its own χ_λ-transform are that $x^{\sigma-1/p}f(x)$ belong to $L^p(0,\infty)$ for $1/p' \leqslant \sigma \leqslant 1/p$, and that*

$$F(s) = \int_0^\infty f(x)x^{s-1}\, dx$$

satisfy the equation $\qquad F(s) = \Omega_\lambda(s)F(1-s)$

when $1/p' < \sigma < 1/p$, and almost everywhere when $\sigma = 1/p$, and $\sigma = 1/p'$. $F(s)$ is an analytic function which is regular in the strip $1/p' < \sigma < 1/p$, it tends uniformly to zero as $|t| \to \infty$ in any interior strip, and it belongs to $L^{p'}(-\infty,\infty)$ as a function of t for $1/p' \leqslant \sigma \leqslant 1/p$.

* *F.I.* Theorems 137 and 138.

† See I. W. Busbridge, *J. of London Math. Soc.* 9 (1934), 179–87, for the case $\lambda = \frac{1}{2}$, $\chi_\lambda(x)$ real. The general case, when $\chi_\lambda(x)$ and λ are complex, is obtained from this by replacing $\Omega(\frac{1}{2}+it)$ by $\Omega_\lambda(\frac{1}{2}+it)$.

SOME ELEMENTARY TAUBERIAN THEOREMS (II)

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In this note I shall prove integral analogues of the theorems for series proved in an earlier paper.* Apart from some minor complications the proofs are straightforward adaptations of arguments used in the 'discontinuous' case. The theorems of the present note represent only one of various possible transcriptions of this kind. The paper concludes with a discussion of a number of examples which show that in some respects the results obtained are not far from the ultimate truth.

We consider the transformation

$$\psi(x) = f(x)\phi(x) + \int_0^\infty g(x,t)\phi(t)\,dt \quad (x \geqslant 0), \tag{1}$$

where $f(x)$, $g(x,t)$ are given functions. To every function $\phi(x)$ corresponds, by means of (1), a function $\psi(x)$, provided that the integral exists. All integrals which occur in this note are supposed to exist as Lebesgue integrals. Functions may have complex values unless it is stated otherwise. By $x \to \infty$ we mean $x \to +\infty$. Relations such as

$$\phi(x) \to 0, \qquad \phi(x) = o\{F(x)\}, \qquad \psi(x) \to 0$$

refer to the limit process $x \to \infty$.

I enumerate a number of hypotheses which I shall use:

$$\varliminf_{x\to\infty} |f(x)| > 0, \tag{2}$$

$$\varlimsup_{x\to\infty} \int_0^\infty \left|\frac{g(x,t)}{f(x)}\right| dt \leqslant \vartheta < 1, \tag{3}$$

$$\lim_{x\to\infty} \frac{g(x,t)}{f(x)} = 0 \quad (t \geqslant 0), \tag{4}$$

$$g(x,t) = 0 \quad (x < t), \tag{5}$$

$$\int_0^\infty \left|\frac{g(x,t)}{f(x)}\right| dt \leqslant \vartheta < 1 \tag{6}$$

* *Quart. J. of Math.* (Oxford), 9 (1938), 274–82.

for every sufficiently large x,

$$g(x,t) = 0 \quad (x+k < t). \tag{7}$$

We say that the function $g(x,t)$ is *majorized* if a function $G(t)$ exists such that

$$|g(x,t)| \leqslant G(t) \quad (x,t \geqslant 0),$$

and

$$\int_0^x G(t)\,dt$$

exists for every positive x. C and x_0 denote given constants and C_1 a suitable constant depending on the data in question.

THEOREM 1. *If* (2), (3), (4) *hold and* $g(x,t)/f(x)$ *is majorized, then*

$$\psi(x) \to 0, \qquad |\phi(x)| \leqslant C \quad (x \geqslant 0)$$

imply

$$\phi(x) \to 0.$$

THEOREM 2. *If* (2), (3), (5) *hold and if* $g(x,t)/f(x)$ *is majorized and* $\phi(x)$ *is bounded in every finite interval, then*

$$|\psi(x)| \leqslant C \quad (x \geqslant x_0)$$

implies

$$|\phi(x)| \leqslant C_1 \quad (x \geqslant 0).$$

THEOREM 3. *If* (2), (3), (4), (5) *hold and if* $g(x,t)/f(x)$ *is majorized and* $\phi(x)$ *is bounded in every finite interval, then* $\psi(x) \to 0$ *implies* $\phi(x) \to 0$.

THEOREM 4. *If* (2), (6), (7) *hold for some positive constant* k, *and if* $\phi(x)$ *is bounded in every finite interval, then*

$$|\psi(x)| \leqslant C \quad (x \geqslant 0),$$

$$\vartheta > 0, \qquad \phi(x) = o(\vartheta^{-x/k})$$

imply

$$|\phi(x)| \leqslant C_1 \quad (x \geqslant 0).$$

THEOREM 5. *If* (2), (4), (6), (7) *hold for some positive constant* k, *and if* $g(x,t)/f(x)$ *is majorized and* $\phi(x)$ *is bounded in every finite interval, then*

$$|\psi(x)| \leqslant C \quad (x \geqslant 0),$$

$$\vartheta > 0, \qquad \psi(x) \to 0, \qquad \phi(x) = o(\vartheta^{-x/k})$$

imply

$$\phi(x) \to 0.$$

THEOREM 6. *Consider a transformation of the form*

$$\psi(x) = f(x)\phi(x)+g(x)\int_0^x h(t)\phi(t)\,dt \quad (x \geqslant 0), \tag{8}$$

where $f(x)$ is real, $\quad \varliminf_{x\to\infty} f(x) > 0, \quad g(x) > 0,$

$$g(x) \leqslant C\,g(x') \quad (x_0 \leqslant x' \leqslant x;\ C \text{ a constant}), \qquad h(t) \geqslant 0 \quad (t \geqslant 0).$$

Then $\varlimsup_{x\to\infty} |\psi(x)| < \infty$ *implies* $\varlimsup_{x\to\infty} |\phi(x)| < \infty$. *If, moreover,* $g(x) \to 0$ *and* $\psi(x) \to 0$, *then* $\phi(x) \to 0$.*

In proving Theorems 1 to 5 we may assume, without loss of generality, that

$$f(x) \neq 0 \quad (x \geqslant 0).$$

For, by (2), this is true for every sufficiently large x, and an alteration of $f(x)$ for small values of x does not affect the validity of our hypotheses. Now, writing (1) in the form

$$\frac{\psi(x)}{f(x)} = \phi(x)+\int_0^\infty \frac{g(x,t)}{f(x)}\phi(t)\,dt,$$

we see at once that in proving Theorems 1 to 5 we may suppose, without loss of generality, that

$$f(x) = 1 \quad (x \geqslant 0).$$

Proof of Theorem 1. Let us assume that

$$0 < \varlimsup_{x\to\infty} |\phi(x)| = r < \infty. \tag{9}$$

Given $\epsilon > 0$, we can find a number $x_0(\epsilon) > 0$ such that, for every $x \geqslant x_0$,

$$|\psi(x)| \leqslant \epsilon, \qquad |\phi(x)| \leqslant r+\epsilon, \qquad \int_0^\infty |g(x,t)|\,dt \leqslant \vartheta+\epsilon.$$

There are arbitrarily large numbers $x_1 > x_0$ for which $|\phi(x_1)| \geqslant r-\epsilon$. Then

$$\begin{aligned} r-\epsilon \leqslant |\phi(x_1)| &= \left|\psi(x_1)-\int_0^\infty g(x_1,t)\phi(t)\,dt\right| \\ &\leqslant |\psi(x_1)|+\left|\int_0^{x_0} g(x_1,t)\phi(t)\,dt\right|+\left|\int_{x_0}^\infty g(x_1,t)\phi(t)\,dt\right| \\ &\leqslant \epsilon+\int_0^{x_0} |g(x_1,t)|C\,dt+(r+\epsilon)(\vartheta+\epsilon). \end{aligned}$$

* In the corresponding theorem for series (loc. cit., Theorem 6) I considered only the case where $g \to 0$, $\psi \to 0$. Of course, the arguments used for proving the present Theorem 6 are at once applicable to series.

As $x_1 \to \infty$ we have, in view of (4), by a fundamental property of the Lebesgue integral ('majorized convergence')

$$\int_0^{x_0} |g(x_1,t)|C\,dt \to 0.$$

Hence $$r-\epsilon \leqslant \epsilon+0+(r+\epsilon)(\vartheta+\epsilon).$$

Making $\epsilon \to 0$, we conclude that $r \leqslant r\vartheta$, i.e. $1 \leqslant \vartheta$. This contradicts (3). Therefore (9) is not true, and the theorem is proved.

Proof of Theorem 2. Suppose that, contrary to the assertion,

$$\overline{\lim_{x\to\infty}}\,|\phi(x)| = \infty. \tag{10}$$

Given $\epsilon > 0$, there is $x_1(\epsilon) > x_0$ such that

$$\int_0^{\infty} |g(x,t)|\,dt \leqslant \vartheta+\epsilon \quad (x \geqslant x_1).$$

For some arbitrary $A > 0$, let $x_2 = x_2(\epsilon, A)$ be the lower bound of all numbers $x \geqslant x_1(\epsilon)$ for which $|\phi(x)| \geqslant A$. Then there are numbers $x_3 \geqslant x_2$, arbitrarily near to x_2, for which $|\phi(x_3)| \geqslant A$. Then

$$\begin{aligned} A \leqslant |\phi(x_3)| &\leqslant |\psi(x_3)|+\left|\int_0^{x_1}\right|+\left|\int_{x_1}^{x_2}\right|+\left|\int_{x_2}^{x_3} g(x_3,t)\phi(t)\,dt\right| \\ &\leqslant C+\int_0^{x_1} G(t)|\phi(t)|\,dt+A(\vartheta+\epsilon)+\int_{x_2}^{x_3} G(t)|\phi(t)|\,dt, \end{aligned}$$

i.e. $$1 \leqslant \frac{C}{A}+\frac{1}{A}\int_0^{x^1} G(t)|\phi(t)|\,dt+\vartheta+\epsilon+\frac{1}{A}\int_{x_2}^{x_3} G(t)|\phi(t)|\,dt.$$

Making $x_3 \to x_2$, we obtain

$$1 \leqslant \frac{C}{A}+\frac{1}{A}\int_0^{x_1} G(t)|\phi(t)|\,dt+\vartheta+\epsilon+0.$$

Now $A \to \infty$ shows that

$$1 \leqslant 0+0+\vartheta+\epsilon,$$

and $\epsilon \to 0$ that $1 \leqslant \vartheta$. This contradiction shows that (10) is not true. Therefore, for suitable constants x_4, C',

$$|\phi(x)| \leqslant C' \quad (x \geqslant x_4).$$

But, by hypothesis,

$$|\phi(x)| \leqslant C'' \quad (0 \leqslant x < x_4).$$

Hence the theorem is proved.

Proof of Theorem 3. Since $\psi(x) \to 0$, we have, for suitable constants, C, x_0,
$$|\psi(x)| \leqslant C \quad (x \geqslant x_0).$$
Therefore, by Theorem 2,
$$|\phi(x)| \leqslant C_1 \quad (x \geqslant 0),$$
and, by Theorem 1,
$$\phi(x) \to 0.$$

Proof of Theorem 4. Without loss of generality, we may assume that
$$\int\limits_0^\infty |g(x,t)|\, dt \leqslant \vartheta \quad (x \geqslant 0).$$
We have
$$\epsilon(x) = \vartheta^{x/k}|\phi(x)| \to 0 \quad \text{as} \quad x \to \infty. \tag{11}$$
Choose any $x_0 \geqslant 0$. We shall show that
$$|\phi(x_0)| \leqslant C(1-\vartheta)^{-1}. \tag{12}$$
Given any $\delta > 0$, we can find x_1 such that
$$0 \leqslant x_1 \leqslant x_0+k,$$
$$|\phi(t)| \leqslant |\phi(x_1)|+\delta \quad (0 \leqslant t \leqslant x_0+k).$$
More generally, if x_{m-1} has been defined already for some $m > 0$, we can find a number x_m such that
$$0 \leqslant x_m \leqslant x_{m-1}+k,$$
$$|\phi(t)| \leqslant |\phi(x_m)|+\delta \quad (0 \leqslant t \leqslant x_{m-1}+k).$$
Then
$$x_m \leqslant x_0+km \quad (m = 0, 1,\ldots). \tag{13}$$
Also, for $\mu = 0, 1,\ldots,$
$$|\phi(x_\mu)| \leqslant |\psi(x_\mu)|+\left|\int\limits_0^{x_\mu+k} g(x_\mu, t)\phi(t)\, dt\right| \leqslant C+\{\delta+|\phi(x_{\mu+1})|\}\vartheta.$$
Using this inequality repeatedly we obtain
$$|\phi(x_0)| \leqslant C+\vartheta\delta+\vartheta|\phi(x_1)| \leqslant C+\vartheta\delta+\vartheta C+\vartheta^2\delta+\vartheta^2|\phi(x_2)| \leqslant \ldots$$
$$\leqslant (C+\vartheta\delta)(1+\vartheta+\vartheta^2+\ldots)+\vartheta^m|\phi(x_m)| \quad (m = 0, 1,\ldots).$$

Case (i). Suppose that $x_m \to \infty$.

Then we conclude, on making use of (11) and (13), that
$$|\phi(x_0)| \leqslant (C+\vartheta\delta)(1-\vartheta)^{-1}+\vartheta^m\vartheta^{-x_m/k}\epsilon(x_m)$$
$$\leqslant (C+\vartheta\delta)(1-\vartheta)^{-1}+\vartheta^m\vartheta^{-(x_0+km)/k}\epsilon(x_m).$$
Therefore, making $m \to \infty$,
$$|\phi(x_0)| \leqslant (C+\vartheta\delta)(1-\vartheta)^{-1}+0.$$
Finally, $\delta \to 0$ shows that (12) holds.

Case (ii). Suppose that, for a suitable constant b,

$$x_m \leqslant b$$

for infinitely many m.

Then, by hypothesis,

$$|\phi(x)| \leqslant C_2 \quad (x \leqslant b).$$

Hence, for infinitely many m,

$$|\phi(x_0)| \leqslant (C+\vartheta\delta)(1-\vartheta)^{-1}+\vartheta^m|\phi(x_m)| \leqslant (C+\vartheta\delta)(1-\vartheta)^{-1}+\vartheta^m C_2.$$

Making $m \to \infty$, we find that

$$|\phi(x_0)| \leqslant (C+\vartheta\delta)(1-\vartheta)^{-1}+0,$$

and again (12) follows by making $\delta \to 0$. This completes the proof.

Proof of Theorem 5. We have, by Theorem 4,

$$|\phi(x)| \leqslant C_1 \quad (x \geqslant 0).$$

Now Theorem 1 is applicable, for (6) implies (3). Therefore $\phi(x) \to 0$.

Proof of Theorem 6. It is no loss of generality to suppose that $\phi(x)$ is a real function and that $\phi(0) = 0$. Corresponding to every $x \geqslant 0$ we define $\bar{x} = \bar{x}(x)$ as being the upper bound of all t such that

$$0 \leqslant t \leqslant x, \qquad \phi(t) \leqslant 0.$$

It is sufficient to prove the following lemma.

LEMMA. *Suppose that the hypotheses of Theorem* 6 *are satisfied. Furthermore, suppose that x tends to infinity through a sequence S which has the property that*

$$f(x) > 0, \qquad \phi(x) \geqslant 0 \quad \textit{for every } x \textit{ in } S,$$

$$\lim_{\substack{x\to\infty \\ x \text{ in } S}} \bar{x}(x) = \bar{\xi} \qquad (\bar{\xi} \textit{ finite or } +\infty).$$

Then
$$\overline{\lim_{\substack{x\to\infty \\ x \text{ in } S}}}\, |\phi(x)| < \infty \qquad \Big[\lim_{\substack{x\to\infty \\ x \text{ in } S}} \phi(x) = 0\Big].^* \tag{14}$$

For, if Theorem 6 were not true, then we should have, for some real function $\phi(x)$ satisfying the hypotheses of the theorem, either

$$\overline{\lim_{x\to\infty}}\, \phi(x) = +\infty \qquad \Big[\overline{\lim_{x\to\infty}}\, \phi(x) > 0\Big] \tag{15}$$

or
$$\underline{\lim_{x\to\infty}}\, \phi(x) = -\infty \qquad \Big[\underline{\lim_{x\to\infty}}\, \phi(x) < 0\Big]$$

* Relations put in square brackets are true under the assumption that $g(x) \to 0$, $\psi(x) \to 0$.

(or both). By multiplying, if necessary, (8) by (-1) and considering $[-\phi(x)]$, $[-\psi(x)]$ instead of $\phi(x)$, $\psi(x)$, we obtain a case where (15) holds. Then we can find a sequence S, tending to infinity, such that

$$\lim_{\substack{x\to\infty \\ x \text{ in } S}} \phi(x) = +\infty \qquad \Big[\lim_{\substack{x\to\infty \\ x \text{ in } S}} \phi(x) = \xi > 0 \quad (\xi \text{ finite or } +\infty)\Big].$$

If we replace S by a suitable sub-sequence, we can obviously satisfy the conditions laid down for S. Then (14) would not be true.

For the rest of this proof x is a number of S, and limiting processes refer to $x \to \infty$ (x in S). Using the definitions of $\bar{x}(x)$, we have, for any number $x' \geqslant 0$,

$$0 \leqslant f(x)\phi(x) = \psi(x) - g(x)\int_0^{x'} - g(x)\int_{x'}^{\bar{x}} - g(x)\int_{\bar{x}}^{x} h(t)\phi(t)\,dt$$

$$\leqslant \psi(x) - \frac{g(x)}{g(x')}[\psi(x') - f(x')\phi(x')] - g(x)\int_{x'}^{\bar{x}} h(t)\phi(t)\,dt - 0.$$

Case (i). Suppose that $\bar{\xi} < \infty$. Then we put $x' = \bar{\xi}$. It follows that

$$\overline{\lim}\, \frac{g(x)}{g(x')}[\psi(x') - f(x')\phi(x')] = \overline{\lim}\, g(x) \times \text{constant} < \infty$$

$$\left[\frac{g(x)}{g(x')}\{\psi(x') - f(x')\phi(x')\} \to 0\right],$$

$$\overline{\lim}\, g(x)\int_{x'}^{\bar{x}} h(t)\phi(t)\,dt < \infty \quad \left[g(x)\int_{x'}^{\bar{x}} h(t)\phi(t)\,dt \to 0\right].$$

Case (ii). Suppose that $\bar{\xi} = \infty$. Then we can determine

$$x' = x'(x) \geqslant 0$$

in such a way that, for sufficiently large x,

$$x_0 - 1 \leqslant \bar{x} - 1 \leqslant x' \leqslant \bar{x}, \qquad \phi(x') \leqslant 0, \qquad \int_{x'}^{\bar{x}} h(t)\phi(t)\,dt \geqslant -1.$$

Then

$$0 \leqslant f(x)\phi(x) \leqslant \psi(x) - \frac{g(x)}{g(x')}\psi(x') + 0 - g(x) \times 1,$$

$$\overline{\lim}\left|\frac{g(x)}{g(x')}\psi(x')\right| \leqslant \overline{\lim}\, C|\psi(x')| < \infty \quad \left[\left|\frac{g(x)}{g(x')}\psi(x')\right| \leqslant C|\psi(x')| \to 0\right].$$

Hence, in either case,

$$\overline{\lim}\, |f(x)\phi(x)| < \infty \quad [f(x)\phi(x) \to 0],$$

$$\overline{\lim}\, |\phi(x)| = \overline{\lim} \frac{|f(x)\phi(x)|}{|f(x)|} < \infty \quad \left[|\phi(x)| = \frac{|f(x)\phi(x)|}{|f(x)|} \to 0\right],$$

and the theorem is proved.

We proceed finally to give the examples mentioned in the introduction.

(i) If $\vartheta = 1$, none of the theorems remains true. For consider the transformation

$$\psi(x) = \begin{cases} \phi(x) & (0 \leqslant x < 1), \\ \phi(x) - \displaystyle\int_{x-1}^{x} \phi(t)\, dt & (1 \leqslant x). \end{cases}$$

Then conditions (2) to (7) are satisfied with $\vartheta = 1$ and arbitrary $k > 0$. If we put $\phi(x) = \sin\sqrt{x}$, then the hypotheses of Theorems 1, 3, 5 are satisfied (with $\vartheta = 1$), but not their conclusions. If we put $\phi(x) = \sqrt{x}$, the same is true with regard to Theorems 2, 4.

(ii) The following example shows that in Theorems 2, 3, 4, 5 it is essential to suppose that $\phi(x)$ is bounded in *every* finite interval (or, in the case of Theorems 4 and 5, that $\psi(x)$ is bounded in the whole range $x \geqslant 0$).

Take the transformation

$$\psi(x) = \begin{cases} \phi(x) & (0 \leqslant x < 1), \\ \phi(x) - \displaystyle\int_{x^{-2}}^{x^{-1}} t^{-\frac{1}{2}}\phi(t)\, dt & (1 \leqslant x). \end{cases}$$

Define ϕ as follows:

$$\phi(x) = \begin{cases} 0 & (x = 0), \\ x^{-\frac{1}{2}} & (0 < x < 1), \\ \log x & (1 \leqslant x). \end{cases}$$

Then the transformed function ψ is given by

$$\psi(x) = \begin{cases} \phi(x) & (0 \leqslant x < 1), \\ 0 & (1 \leqslant x). \end{cases}$$

Here (2), (3) (with $\vartheta = 0$), (4), (5), (6) (with any $\vartheta > 0$) are satisfied, $g(x,t)/f(x)$ is majorized, $\psi(x) \to 0$, $\psi(x)$ is bounded in every interval $\delta \leqslant x < \infty$, and $\phi(x)$ is bounded in every interval $\delta \leqslant x \leqslant b$ where δ is any positive constant, and $\phi(x) = o\,(\vartheta^{-x/k})$ for any positive ϑ, k. But $\phi(x) \to \infty$ as $x \to \infty$.

Similarly, the transformation

$$\psi(x) = \begin{cases} \phi(x) & (0 \leqslant x < 2), \\ \phi(x) - \int\limits_{x^{-1}}^{2x^{-1}} t^{-\frac{1}{2}}\phi(t)\,dt & (2 \leqslant x), \end{cases}$$

if we substitute for ϕ the function

$$\phi(x) = \begin{cases} 0 & (x = 0), \\ x^{-\frac{1}{2}} & (0 < x < 1), \\ \log 2 & (1 \leqslant x), \end{cases}$$

shows that also in Theorem 1 it is essential to suppose that ϕ or ψ is bounded in *every* finite interval.

(iii) In Theorems 4 and 5 we cannot replace (6) by (3). For consider the transformation

$$\psi(x) = \phi(x) - \vartheta \frac{x+k+1}{x+k} \frac{1}{\delta(x)} \int\limits_{x+k-\delta(x)}^{x+k} \phi(t)\,dt,$$

where ϑ, k are positive constants, $\vartheta < 1$, and $\delta(x)$ is a function of x satisfying $0 < \delta(x) < k$ which will be specified later. Put

$$\phi(x) = \vartheta^{-x/k}(x+1)^{-1} \quad (x \geqslant 0).$$

Then

$$\psi(x) = \phi(x) - \vartheta(x+k+1)(x+1)^{-1}\phi\{x+k-\delta_1(x)\} \text{ for some } \delta_1(x),$$

$0 \leqslant \delta_1(x) \leqslant \delta(x)$. Hence

$$\psi(x) = \phi(x)[1 - \vartheta^{\delta_1/k}(x+k+1)(x+k+1-\delta_1)^{-1}] \to 0$$

as $x \to \infty$, provided $\delta(x)$ tends to zero sufficiently rapidly. Then (3) holds as well as all hypotheses of Theorems 4 and 5 with the exception of (6). But the conclusions of these theorems do not hold.

(iv) In Theorems 4 and 5 the condition

$$\phi(x) = o(\vartheta^{-x/k})$$

cannot be replaced by $\quad \phi(x) = O(\vartheta^{-x/k}).$

For consider the transformation

$$\psi(x) = \phi(x) - \frac{\vartheta}{\rho} \int\limits_{x+k-\rho}^{x+k} \phi(t)\,dt \quad (x \geqslant 0),$$

where $\qquad 0 < \vartheta < 1, \qquad 0 < k, \qquad \rho = k\vartheta^{2x/k}.$

If
$$\phi(x) = \vartheta^{-x/k} \quad (x \geqslant 0),$$
then
$$\psi(x) = \vartheta^{-x/k} - \frac{1}{\rho} \int\limits_{x+k-\rho}^{x+k} \vartheta^{(-t+k)/k}\, dt = \vartheta^{-x/k} - \vartheta^{-x_1/k}$$
for some x_1 $(x-\rho \leqslant x_1 \leqslant x)$. Therefore, by the mean-value theorem,
$$|\psi(x)| = \left|\frac{-x+x_1}{k} \vartheta^{-x_2/k} \log \vartheta\right| \quad (x-\rho \leqslant x_1 \leqslant x_2 \leqslant x)$$
$$\leqslant \frac{\rho}{k} \vartheta^{-x/k} \log \vartheta = \vartheta^{x/k} \log \vartheta \to 0$$
as $x \to \infty$.

FINITE SUMMATION FORMULAE

By A. P. GUINAND (*Oxford*)

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1. Introduction

It has recently been shown that for certain classes of sequences $\{a_n\}$ and functions $f(x)$ there exist summation formulae connecting sums of the form

$$\sum_{n=1}^{\infty} a_n f(n) \tag{1.1}$$

with corresponding sums of the form

$$\sum_{n=1}^{\infty} a_n \int_0^{\infty} f(x)\, K(nx)\, dx, \tag{1.2}$$

where $K(x)$ is a Fourier kernel.*

A number of examples of summation formulae connecting infinite series of the forms (1.1) and (1.2) are known. In this paper I derive some summation formulae associated with finite sums of the form

$$\sum_{n=1}^{N} a_n f(n).$$

These summation formulae and the corresponding inversion formulae are of a particularly simple character, and can be proved by direct methods.

In the last section some self-reciprocal functions associated with these summation formulae are discussed.

2. Formalities

If we apply the general theorem† for summation formulae to a finite sequence $a_1, a_2,..., a_N$ we find that, with appropriate conditions,

$$\sum_{n=1}^{N} a_n f(n) = \sum_{n=1}^{N} a_n\, g(n),$$

where

$$g(x) = \frac{d}{dx} \int_0^{\infty} f(y) \frac{\chi(xy)}{y}\, dy,$$

$$\chi(x) = \underset{T\to\infty}{\mathrm{l.i.m.}} \frac{1}{2\pi i} \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \frac{\psi(1-s)}{\psi(s)} \frac{x^{1-s}}{1-s}\, ds, \tag{2.1}$$

* W. L. Ferrar, *Compositio Math.* 4 (1937), 394–405; A. P. Guinand, *Quart. J. of Math.* (Oxford), 9 (1938), 53–67.

† A. P. Guinand, loc. cit., Theorem 2.

and $$\psi(s) = \sum_{n=1}^{N} a_n n^{-s}. \tag{2.2}$$

Let us assume that

$$\frac{\psi(1-s)}{\psi(s)} = \left\{\sum_{n=1}^{N} a_n n^{s-1}\right\}\left\{\sum_{n=1}^{N} a_n n^{-s}\right\}^{-1} = \sum_{n=1}^{\infty} b_n c_n^{s-1} \tag{2.3}$$

when expanded by the multinomial theorem, and that the series on the right is absolutely convergent for $s = \frac{1}{2}$.

For instance, if

$$a_1 = 1, \qquad \sum_{n=2}^{N} |a_n| n^{-\frac{1}{2}} < 1, \qquad R(s) = \tfrac{1}{2},$$

the expression becomes

$$\left\{\sum_{n=1}^{N} a_n n^{s-1}\right\}\left\{1 + \sum_{n=2}^{N} a_n n^{-s}\right\}^{-1}$$

$$= \left\{\sum_{n=1}^{N} a_n n^{s-1}\right\}\left\{\sum_{m=0}^{\infty} (-)^m \left(\sum_{n=2}^{N} a_n n^{-s}\right)^m\right\}$$

$$= \sum_{n=1}^{N} a_n \sum_{m=0}^{\infty} (-)^m m! \sum_{\alpha+\beta+\ldots+\omega=m} \frac{a_2^{\alpha} a_3^{\beta} \ldots a_N^{\omega}}{\alpha!\,\beta! \ldots \omega!\, 2^{\alpha} 3^{\beta} \ldots N^{\omega}} \left(\frac{n}{2^{\alpha} 3^{\beta} \ldots N^{\omega}}\right)^{s-1},$$

where the number of the $\alpha, \beta, \ldots, \omega$ is $N-1$. This series is of the required form (2.3). In other cases the expansion will be more complicated.

Substituting (2.3) in (2.1) we have, formally,

$$\chi(x) = \underset{T\to\infty}{\text{l.i.m.}} \frac{1}{2\pi i} \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \sum_{n=1}^{\infty} b_n \left(\frac{x}{c_n}\right)^{1-s} \frac{ds}{1-s}$$

$$= \sum_{n=1}^{\infty} \lim_{T\to\infty} \frac{b_n}{2\pi i} \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \left(\frac{x}{c_n}\right)^{1-s} \frac{ds}{1-s}$$

$$= \sum_{n=1}^{\infty} \chi_n(x),$$

where $$\chi_n(x) = \begin{cases} 0 & (x < c_n), \\ \frac{1}{2} b_n & (x = c_n), \\ b_n & (x > c_n). \end{cases}$$

Hence

$$g(x) = \frac{d}{dx}\int_0^\infty \frac{f(y)}{y}\sum_{n=1}^\infty \chi_n(xy)\,dy$$

$$= \sum_{n=1}^\infty b_n \frac{d}{dx}\int_{c_n/x}^\infty \frac{f(y)}{y}\,dy$$

$$= \sum_{n=1}^\infty b_n\left(-\frac{c_n}{x^2}\right)\left\{-\frac{x}{c_n}f\left(\frac{c_n}{x}\right)\right\}$$

$$= \sum_{n=1}^\infty \frac{b_n}{x}f\left(\frac{c_n}{x}\right)$$

formally.

3. The finite summation formula

THEOREM 1. *If* (i) *there is a finite sequence of real numbers* $a_1, a_2,\ldots, a_N$ *such that the series* (2.3) *is absolutely convergent for* $s = \frac{1}{2}$ *and*

$$\sum_{n=1}^\infty |b_n|$$

is convergent, and

(ii) $$|f(x)| < K \quad \textit{for all } x,$$

then $$\sum_{n=1}^N a_n f(n) = \sum_{n=1}^N a_n\, g(n), \tag{3.1}$$

where $$g(x) = \sum_{n=1}^\infty \frac{b_n}{x} f\left(\frac{c_n}{x}\right). \tag{3.2}$$

Further $$f(x) = \sum_{n=1}^\infty \frac{b_n}{x} g\left(\frac{c_n}{x}\right). \tag{3.3}$$

From (2.3), for $R(s) = \frac{1}{2}$,

$$\sum_{n=1}^N a_n\, n^{s-1} = \left\{\sum_{n=1}^N a_n\, n^{-s}\right\}\left\{\sum_{m=1}^\infty b_m c_m^{s-1}\right\}$$

$$= \sum_{n=1}^N \sum_{m=1}^\infty a_n\, n^{-1} b_m (c_m\, n^{-1})^{s-1}.$$

Hence, by the uniqueness theorem for Dirichlet's series,*

$$\sum_{c_m/n=r} a_n\, n^{-1} b_m$$

* E. C. Titchmarsh, *Theory of Functions* (Oxford, 1932), 309. The result for the general Dirichlet's series $\sum_{n=1}^\infty a_n e^{-\lambda_n s}$ is proved in the same way.

converges to a_r if r is a positive integer not greater than N, and to zero for other values of r. Also the series (3.2) converges absolutely, since

$$\sum_{n=1}^{\infty} \left|\frac{b_n}{x} f\left(\frac{c_n}{x}\right)\right| < \frac{K}{|x|} \sum_{n=1}^{\infty} |b_n|.$$

Hence $$\sum_{n=1}^{N} a_n g(n) = \sum_{n=1}^{N} \sum_{m=1}^{\infty} a_n n^{-1} b_m f(c_m/n) = \sum_{r=1}^{N} a_r f(r).$$

From (2.3)

$$\left\{\sum_{m=1}^{\infty} b_m c_m^{s-1}\right\}\left\{\sum_{n=1}^{\infty} b_n c_n^{-s}\right\} = 1 \quad (R(s) = \tfrac{1}{2}).$$

Hence $$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_m b_n c_m^{-1} (c_m/c_n)^s = 1,$$

and, by the uniqueness theorem for Dirichlet's series,

$$\sum_{c_m/c_n=\alpha} b_m b_n c_m^{-1} = \begin{cases} 0 & (\alpha \neq 1), \\ 1 & (\alpha = 1). \end{cases}$$

Hence, by (3.2),

$$\sum_{m=1}^{\infty} \frac{b_m}{x} g\left(\frac{c_m}{x}\right) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_m b_n c_m^{-1} f\left(\frac{c_n x}{c_m}\right) = f(x);$$

this completes the proof of the theorem.

4. Examples

(i) If $$g(x) = \frac{N}{x} f\left(\frac{N^2}{x}\right),$$

then $$f(N) = g(N),$$

and $$f(x) = \frac{N}{x} g\left(\frac{N^2}{x}\right).$$

(ii) If $$|Mb| < |Na|, \qquad |Mb^2| < |Na^2|,$$

and

$$g(x) = \frac{Mb}{ax} f\left(\frac{MN}{x}\right) + \frac{M}{Na^2 x}(Na^2 - Mb^2) \sum_{n=0}^{\infty} \left(-\frac{Mb}{Na}\right)^n f\left(\frac{M^{n+2}}{N^n x}\right),$$

then $$af(M) + bf(N) = ag(M) + bg(N),$$

and the corresponding inversion formula for $f(x)$ holds.

(iii) If

$$g(x) = \frac{1}{x}\sum_{n=0}^{\infty}\left\{(-3)^{-n}f\left(\frac{2}{3^n x}\right)+(-2)^{-n}f\left(\frac{3}{2^n x}\right)\right\}+$$
$$+\frac{1}{x}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-)^{m+n}(m+n+1)!}{m!\,n!\,2^m 3^n}\times$$
$$\times\left\{\frac{1}{m+n+1}-\frac{1}{2(m+1)}-\frac{1}{3(n+1)}\right\}f\left(\frac{1}{2^m 3^n x}\right),$$

then
$$f(1)+f(2)+f(3) = g(1)+g(2)+g(3),$$
and the corresponding inversion formula for $f(x)$ holds.

(iv) Similar results hold for unsymmetrical summation formulae. For example, if

$$g(x) = \frac{1}{x}\sum_{n=0}^{\infty}(-2)^{-n}f\left(\frac{1}{2^n x}\right),$$

then
$$f(1) = g(1)+g(2),$$
and
$$f(x) = \frac{1}{x}g\left(\frac{1}{x}\right)+\frac{1}{x}g\left(\frac{2}{x}\right).$$

5. Self-reciprocal functions

I have shown* that to a certain class of summation formulae there corresponds a class of self-reciprocal functions. The corresponding result for the finite summation formula of Theorem 1 is

$$\int_0^x F(y)\,dy = \int_0^\infty F(y)\frac{\chi_1(xy)}{y}\,dy,$$

where
$$F(x) = \begin{cases} x^{-1}\sum\limits_{1\leqslant n\leqslant x}{}' a_n & (x \leqslant N), \\ x^{-1}\sum\limits_{n=1}^{N} a_n & (x > N), \end{cases} \tag{5.1}$$

and
$$\chi_1(x) = \int_0^x \chi(u)u^{-1}\,du - \chi(x).$$

If we consider the particular case $N = a_1 = 1$, we have

$$\chi_1(x) = \begin{cases} 0 & (x<1), \\ \int\limits_1^x u^{-1}\,du - 1 = \log x - 1 & (x>1), \end{cases}$$

* A. P. Guinand, loc. cit., Theorem 1, and *Proc. London Math. Soc.* (2), 43 (1937), 439–48.

and we are led to an inversion formula

$$g(x) = \frac{d}{dx}\int_{1/x}^{\infty} y^{-1}f(y)(\log xy - 1)\,dy$$

$$= \frac{1}{x}\int_{1/x}^{\infty} \frac{f(y)}{y}\,dy - \frac{1}{x}f\left(\frac{1}{x}\right). \tag{5.2}$$

The inverse formula can be proved directly.* We have

$$\frac{1}{x}\int_{1/x}^{\infty} \frac{g(t)}{t}\,dt - \frac{1}{x}g\left(\frac{1}{x}\right)$$

$$= \frac{1}{x}\int_{1/x}^{\infty} \frac{dt}{t^2}\int_{1/t}^{\infty} \frac{f(y)}{y}\,dy - \frac{1}{x}\int_{1/x}^{\infty} f\left(\frac{1}{t}\right)\frac{dt}{t^2} - \int_{x}^{\infty} \frac{f(y)}{y}\,dy + f(x)$$

$$= \frac{1}{x}\int_{0}^{x} dz \int_{z}^{\infty} \frac{f(y)}{y}\,dy - \frac{1}{x}\int_{0}^{x} f(z)\,dz - \int_{x}^{\infty} \frac{f(y)}{y}\,dy + f(x)$$

$$= \frac{1}{x}\int_{0}^{x} \frac{f(y)}{y}\,dy \int_{0}^{y} dz + \frac{1}{x}\int_{x}^{\infty} \frac{f(y)}{y}\,dy \int_{0}^{x} dz -$$

$$- \frac{1}{x}\int_{0}^{x} f(z)\,dz - \int_{x}^{\infty} \frac{f(y)}{y}\,dy + f(x) = f(x).$$

The argument is readily justified if we assume that

$$\int_{0}^{\infty} |f(t)|\,\frac{dt}{t}$$

converges. Further (5.1) reduces to

$$F(x) = \begin{cases} 0 & (x < 1), \\ \frac{1}{2} & (x = 1), \\ x^{-1} & (x > 1), \end{cases}$$

* An inversion formula of a similar type involving Bessel functions has been given by E. C. Titchmarsh, *Theory of Fourier Integrals* (Oxford, 1937), 219 (11).

and

$$\frac{1}{x}\int_{1/x}^{\infty}\frac{F(y)}{y}\,dy-\frac{1}{x}F\left(\frac{1}{x}\right)=\begin{cases}\dfrac{1}{x}\displaystyle\int_{1/x}^{\infty}\frac{dy}{y^2}-1=0 & (x<1),\\ \displaystyle\int_{1}^{\infty}\frac{dy}{y^2}-\tfrac{1}{2}=\tfrac{1}{2} & (x=1),\\ \dfrac{1}{x}\displaystyle\int_{1}^{\infty}\frac{dy}{y^2}=x^{-1} & (x>1),\end{cases}$$

i.e. $F(x)$ is self-reciprocal with respect to the transformation (5.2). In the same way we can prove the result for the general case. The result is:

THEOREM 2. *If, in addition to the assumptions of Theorem* 1,

$$\int_{0}^{\infty}|f(t)|\,\frac{dt}{t}$$

converges, and

$$g(x)=\frac{1}{x}\sum_{n=1}^{\infty}b_n\left\{\int_{c_n/x}^{\infty}\frac{f(y)}{y}\,dy-f\left(\frac{c_n}{x}\right)\right\},$$

then

$$f(x)=\frac{1}{x}\sum_{n=1}^{\infty}b_n\left\{\int_{c_n/x}^{\infty}\frac{g(y)}{y}\,dy-g\left(\frac{c_n}{x}\right)\right\}.$$

Further, the function $F(x)$ defined by (5.1) *is self-reciprocal with respect to this transformation.*

WURZELN AUS DER HANKEL-, FOURIER- UND AUS ANDEREN STETIGEN TRANSFORMATIONEN

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1. SEI $\mathfrak{L}_2(a,b)$ $(-\infty \leqslant a < b \leqslant \infty)$ der Hilbertsche Raum, der aus allen komplexwertigen, quadratisch von a bis b Lebesgue-integrierbaren Funktionen besteht, mit dem 'inneren Produkte'

$$(f,g) = \int_a^b f(x)\overline{g(x)}\,dx, \qquad f \in \mathfrak{L}_2(a,b), \quad g \in \mathfrak{L}_2(a,b),$$

sei $\|f\| = (f,f)^{\frac{1}{2}} \geqslant 0$; $f_n \sim f$ bedeute $\lim \|f-f_n\| = 0$ für $n \to \infty$, $f(x) \sim a_0\phi_0+a_1\phi_1+a_2\phi_2+\ldots$ oder $f \sim \sum\limits_{n=0}^{\infty} a_n\phi_n$ bedeute

$$\|f-\sum_{n=0}^{N} a_n\phi_n\| \to 0 \text{ für } N \to \infty,$$

$((u))$ die Differenz zwischen u und der nächsten ganzen Zahl, $2u$ nicht ganz, also $|((u))| < \frac{1}{2}$.

Die Transformation $g = T_r f$ im Raume $\mathfrak{L}_2(0,\infty)$

$$g(x) = T_r^{(\alpha)} f = c_r^{(\alpha)} \underset{n\to\infty}{\mathrm{l.i.m.}} \int_0^n J_\alpha\!\left(\frac{xy}{|\sin \pi r|}\right)(xy)^{\frac{1}{2}} e^{-\frac{1}{2}i(x^2+y^2)\cot \pi r} f(y)\,dy, \tag{1}$$

$$c_r^{(\alpha)} = |\sin \pi r|^{-1} \exp\{i\pi((\tfrac{1}{2}-r))(1+\alpha)\}$$

$$(\Re(\alpha) > -1;\ r \text{ reell nicht ganz})$$

ist stetig, für reelles α unitär und für $r \equiv \frac{1}{2} \pmod 1$ die gewöhnliche Hankel-Transformation $T^{(\alpha)}$, für $r = \frac{1}{4}, \frac{1}{6}, \ldots, 1/2m$ z. B. eine Transformation, deren zweite, dritte,..., m. Potenz $T^{(\alpha)}$ ist. Entsprechend ist die Transformation

$$g(x) = T_r f = C_r \underset{n\to\infty}{\mathrm{l.i.m.}} \int_{-n}^{n} \exp\!\left(\frac{itx}{\sin 2\pi r} - i\frac{x^2+t^2}{2}\cot 2\pi r\right) f(t)\,dt, \tag{2}$$

$$C_r = (2\pi|\sin 2\pi r|)^{-\frac{1}{2}} e^{\frac{1}{2}i\pi((\frac{1}{2}-2r))} \quad (r \text{ reell; } 2r \text{ nicht ganz})$$

unitär in $\mathfrak{L}_2(-\infty,\infty)$ und für $r \equiv \frac{1}{4} \pmod 1$ die gewöhnliche Fourier-Transformation F, für $r = 1/4m$ z. B. eine Transformation, deren m. Potenz F ist, also eine 'm. Wurzel' aus F. Es gehören (1) für reelles α $(\alpha > -1)$ und (2) zu je einer Schar von Transformationen folgender Art:

SATZ I.* *Sei* $\{\phi_n\}$ $(n = 0, 1, ...)$ *ein vollständiges orthonormales (d. h. orthogonales und normiertes) System im Hilbertschen Raume* $\mathfrak{H}$, r reell,

$$T_r f = g \sim (f, \phi_0)\phi_0 + e^{2i\pi r}(f, \phi_1)\phi_1 + ... = \sum_{n=0}^{\infty} e^{2i\pi rn}(f, \phi_n)\phi_n; f \in \mathfrak{H}. \quad (3)$$

Diese Transformation $g = T_r f$ *hat folgende Eigenschaften:*

(*a*) T_r *ist unitär in* $\mathfrak{H}$, *die Umkehrung ist* T_{-r}.

(*b*) $T_{r+1} = T_r$; T_0 *ist die identische Transformation* I.

(*c*) (α) $T_s(T_r f) = T_{r+s} f = T_r(T_s f)$. ($\beta$) $T_r^m f = T_{mr} f$ *für jedes ganze* m.

(*d*) *Die sämtlichen Eigenwerte von* T_r *sind* $\exp(2i\pi rk)$ $(k = 0, 1, ...)$, *für irrationales* r *sämtlich verschieden, zu jedem gehört dann genau eine bis auf eine Konstante bestimmte Eigenfunktion* $a\phi_k$.

(*e*) *Für rationales* $r = n/m$ *(*m *und* n *ganz und teilerfremd):*

(α) *gibt es genau* m *Eigenwerte,* (β) *bilden die zum* k. *Eigenwerte* $e^{2i\pi rk}$ $(0 \leqslant k < m)$ *gehörigen Eigenfunktionen unter Hinzunahme des Elementes Null einen Hilbertschen Raum* $\mathfrak{R}_k$ *mit dem vollständigen orthonormalen System* $\{\phi_{k+sm}\}$ $(s = 0, 1, ...)$, (γ) *ist jede Funktion aus* $\mathfrak{H}$ *eindeutig als Summe von höchstens* m *zu verschiedenen Eigenwerten gehörigen Eigenfunktionen darstellbar,* (δ) *ist* $\mathfrak{R}_k$ *orthogonal zu* $\mathfrak{R}_l$ *für* $k \neq l$, (ϵ) *ist* $T_r^m = T_0 = I$.

(*f*) *Für* $s \to r$ *ist* $\lim T_s = T_r$, *d.h.* $T_s f \rightharpoonup T_r f$ *für jedes* f *aus* $\mathfrak{H}$.

Die analytische Darstellung der Schar T_r führt mit Benutzung der Laguerreschen bezw. Hermiteschen Polynome auf (1) bezw. (2) und erfolgt mit Hilfe der klassisch gewordenen Methode Plancherels, von Wienerschen und Bochnerschen Gedanken und durch Weiterführung dieser Überlegungen zu dem allgemeinen Satze II (Abschn. 3).

Wählt man $\mathfrak{H} = \mathfrak{L}_2(0, 1)$, $\phi_0 = 1$, $\phi_n = \sqrt{2}\cos \pi nx$ $(n = 1, 2, ...)$, so erhält man eine weitere einfache Schar T_r (Abschn. 6).

Satz I gilt für unitäre Scharen T_r. Der allgemeinere Fall stetiger, nicht notwendig unitärer Transformationen, zu denen (1) für komplexes α gehört, verlangt ein abgeändertes Beweisverfahren, das System $\{\phi_n\}$ ist dann nur als 'abgeschlossen in $\mathfrak{H}$'† vorauszusetzen (Abschn. 7 bis 10).

* Eigenwerte sind hier im Sinne der 'Hilbert-Raum'-Definition gemeint, also Zahlen l, für welche $Tf - lf = 0$ eine Lösung $f \not\equiv 0$ hat, s. M. H. Stone, *Linear Transformations in Hilbert Space and their Applications to Analysis* (New York, 1932), chapter 4.

† S. Kaczmarz und H. Steinhaus, *Theorie der Orthogonalreihen* (Warszawa-Lwòw, 1935), Def. [241] und Satz [354], [355]. Hier sind aber $f(x)$, $\phi(x)$ u.s.w. komplexwertig.

Eine Schar unitärer Transformationen mit den Gruppen- und Stetigkeitseigenschaften ($c(\alpha)$) und (f) ist eine 'Stonesche',* symbolisch darstellbar in der Form $T_r \equiv e^{irH}$; Hf ist eine selbstadjungierte, hier die (unstetige) Transformation $Hf \sim 2\pi \sum n(f, \phi_n)\phi_n$. Die Schar $A_z f \sim \sum e^{-2\pi nz}(f, \phi_n)\phi_n$ ($\Re(z) > 0$) ist eine 'Hillesche Halbgruppe'† und durch $A_z \equiv e^{-zH}$ darstellbar. Eine Stonesche Schar kann dann und nur dann, wenn H positiv definit ist, in eine Hillesche Halbgruppe fortgesetzt werden ($r = iz$, (f) gilt dann für $\Re(z) \geqslant 0$), umgekehrt ist jede Hillesche unter gewissen Bedingungen die Fortsetzung einer Stoneschen Schar.—Herrn Hille für seine Anregungen besten Dank!

2. *Beweis des Satzes I* (a). Da jede Folge $\{a_n\}$ ($n = 0, 1, \ldots$) mit $\sum |a_n|^2 < \infty$, genau eine Funktion $h \sim \sum a_n \phi_n$ bestimmt, $h \in \mathfrak{H}$, $(h, \phi_n) = a_n$, und umgekehrt, wird durch $g = T_r f$ jedem $f \sim \sum a_n \phi_n$ genau ein $g \sim \sum a_n e^{2i\pi rn}\phi_n$, jedem $g \sim \sum b_n \phi_n$ daher umgekehrt genau ein $f \sim \sum b_n e^{-2i\pi rn}\phi_n \sim T_{-r} g$ zugeordnet; wegen

$$(g_1, g_2) = \sum b_n^{(1)} \overline{b_n^{(2)}} = \sum a_n^{(1)} \overline{a_n^{(2)}} = (f_1, f_2)$$

ist also T_r unitär (cf. Stone, Theor. 1.9 (5), Def. 2. 18).

Satz I (b) folgt aus $\exp\{2i\pi(r+1)\} = \exp(2i\pi r)$ und $\exp(2i\pi 0) = 1$.

Der Beweis von I (c) und I (e) ist unschwer unter Benutzung der bei den anderen Punkten gebrauchten Hilfsmittel zu führen; doch folgen I (c) und (e) sofort aus den Punkten I′ (c) und (e) des allgemeinen Satzes I′ (Abschn. 7), da ein in einem Hilbertschen Raume abgeschlossenes System, wenn es dazu noch orthonormal ist, ein vollständiges orthonormales System in diesem Raume ist.

I (d). Sei nun $\eta_r = e^{2i\pi r}$. Für $f = \phi_k$ ist, da $\{\phi_n\}$ ein vollständiges orthonormales System ist, $g \sim \sum \eta_r^n(\phi_k, \phi_n)\phi_n = \eta_r^k \phi_k$, also ist η_r^k Eigenwert ($k = 0, 1, \ldots$). Dies sind sämtliche Eigenwerte: Wäre η von allen η_r^k verschieden, ψ eine zu η gehörige Eigenfunktion, also $T_r \psi = \eta\psi$, so würde aus $\psi \sim \sum c_n \phi_n$ folgen: $T_r \psi \sim \sum c_n \eta_r^n \phi_n$, hieraus

$$\sum c_n(\eta_r^n - \eta)\phi_n = \sum c_n \eta_r^n \phi_n - \eta \sum c_n \phi_n \sim T_r \psi - \eta\psi = 0, \quad (2.1)$$

also $c_n = 0$ und der Widerspruch $\psi = 0$.

Ist r irrational, so sind alle η_r^k von einander verschieden; sei F_k irgendeine zu η_r^k gehörige Eigenfunktion, also $T_r F_k = \eta_r^k F_k$, so folgt

* *Annals of Math.* 2, 33 (1932), 643–8.

† *Proc. Nat. Acad. Sci.* 24 (1938), 159–61.

aus (2.1) für $\eta = \eta_r^k$, dass alle c_n Null sind bis auf c_k, also $F_k = c_k \phi_k$. Ersichtlich lässt sich jedes $f \in \mathfrak{H}$ eindeutig in eine Reihe von Eigenfunktionen entwickeln.

I (f). Für $f \sim \sum a_n \phi_n$ ist $T_r f \sim \sum a_n e^{2i\pi rn}\phi_n$, also

$$\|T_s f - T_r f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 |e^{2i\pi sn} - e^{2i\pi rn}|^2 = \sum_0^U + \sum_U^\infty.$$

Sei $\epsilon > 0$. Wegen $\sum |a_n|^2 < \infty$ und $|e^{2i\pi rn} - e^{2i\pi sn}| \leqslant 2$ kann man U so festlegen, dass die letzte Summe kleiner als $\frac{1}{2}\epsilon^2$, dann $\delta > 0$ so, dass die vorletzte Summe für $|r-s| < \delta$ ebenfalls kleiner als $\frac{1}{2}\epsilon^2$ wird, also $\|T_s f - T_r f\| < \epsilon$; folglich $T_s f \sim T_r f$ für $s \to r$.

Für irrationales r kann man daher Folgen von ganzen Zahlen $m_1, m_2, \ldots$ bestimmen derart, dass $T_r^{m_\nu} f \sim f$ wird für $m_\nu \to \infty$.

Es gibt bei jedem Grade m unendlich viele Wurzeln aus einer Transformation T_r. Sei r fest, $\eta^{(n)}$ $(n = 0, 1, \ldots)$, z. B. eine Folge, die aus den Zahlen $\pm e^{i\pi rn}$ bei beliebiger Anordnung der Vorzeichen besteht, also $(\eta^{(n)})^2 = e^{2i\pi rn}$, dann ist für jede Transformation $Sf \sim \sum \eta^{(n)}(f, \phi_n)\phi_n$ ersichtlich $S^2 f \sim \sum (\eta^{(n)})^2 (f, \phi_n)\phi_n \sim T_r f$.

3.* Sei $\mathfrak{u}(x)$ die Treppenfunktion, die gleich Eins für $0 \leqslant x \leqslant 1$, sonst Null ist, $\{\phi_n\}$ $(n = 0, 1, \ldots)$ ein vollständiges orthonormales System in $\mathfrak{L}_2(a, b)$, $a \leqslant 0 \leqslant b$, daher für $a \leqslant z \leqslant b$

$$\mathfrak{u}(x/z) \sim \sum_0^\infty \phi_n(x)\big(\mathfrak{u}(x/z), \phi_n(x)\big) = \operatorname{sgn} z \sum_0^\infty \phi_n(x) \int_0^z \overline{\phi_n(y)}\, dy, \quad (3.1)$$

$$\sum_0^\infty \left| \int_0^z \phi_n(y)\, dy \right|^2 = \|\mathfrak{u}(x/z)\|^2 = |z|. \quad (3.11)$$

Statt (3) betrachte man die allgemeinere Transformation

$$g = T^{(t)} f \sim \sum t^n (f, \phi_n)\phi_n, \qquad |t| \leqslant 1; f \in \mathfrak{L}_2(a, b), \quad (3')$$

die wegen $\|g\| \leqslant \|f\|$ zwar stetig ist, aber für $|t| < 1$ nicht unitär ist und sogar nur eine unstetige Umkehrung besitzt.†

Bekanntlich ist die doppelt unendliche Folge $\{\overline{\phi_m(x)}\phi_n(y)\}$ ein voll-

* Zu diesem Abschnitt: M. Plancherel, *Rend. di Palermo*, 30 (1910), 289–335, und *Comment. Math. Helvetici*, 9 (1937), 249–62, bes. 253–7; N. Wiener, *The Fourier Integral* (Cambridge, 1933); S. Bochner, *Annals of Math.* 35 (1934), 111–15.

† Für $g = \phi_n$ ist $f = t^{-n}\phi_n$, $\|f\|/\|g\| = |t|^{-n} \to \infty$ für $n \to \infty$; k ist ein Kern vom 'Hilbert-Schmidt' Typ.

ständiges orthonormales System in $\mathfrak{L}_2^{(2)}(a,b)$,* also ist für $|t|<1$ wegen $\sum |t^n|^2 < \infty$

$$k(x,y;t) \sim \sum t^n \overline{\phi_n}(x)\phi_n(y) \tag{3.2}$$

eine in $\mathfrak{L}_2^{(2)}(a,b)$ bis auf eine zweidimensionale Menge vom Masze Null wohldefinierte Funktion; für fast alle y gehört daher $k(x,y;t)$ als Funktion von x auch zu $\mathfrak{L}_2(a,b)$, nach Parsevals Satz ist für fast alle y in (a,b)

$$\int_a^b k(x,y;t)f(x)\,dx = \big(f(x),\overline{k(x,y;t)}\big) = \sum_{n=0}^{\infty} (f,\phi_n)t^n\phi_n(y), \tag{3.21}$$

also für $|t|<1$ fast überall

$$T^{(t)}f = g(z) = \int_a^b k(\xi,z;t)f(\xi)\,d\xi. \tag{3.3}$$

Man wähle jetzt $a \leqslant 0$ endlich, $b=\infty$. Es gilt

SATZ II. (i) *Sei τ eine feste Zahl vom Absolutwerte Eins.*

(ii) *Für eine Zahlenfolge $\{t_n\}$, $|t_n|<1$, $t_n \to \tau$, möge $k(x,y;t_n)$ (s. 3.2) gegen eine Funktion $k(x,y;\tau)$ fast überall in $a \leqslant x < \infty$, $a \leqslant y < \infty$ streben.*

(iii) *Gleichmässig für die Folge $\{t_n\}$ sei $|k(x,y;t_n)| \leqslant F(x,y)$, und die Funktion F möge für jedes $A>0$ zu $\mathfrak{L}_2^{(2)}(a,A)$ gehören.*

Dann ist die Transformation $g = T^{(\tau)}f \sim \sum \tau^n (f,\phi_n)\phi_n$ gleichwertig mit jeder der beiden Gleichungen

$$\int_0^v g(\xi)\,d\xi = \int_a^\infty d\xi\, f(\xi) \int_0^v k(\xi,x;\tau)\,dx, \tag{3a}$$

$$g(x) = \underset{n\to\infty}{\mathrm{l.i.m.}} \int_a^n k(\xi,x;\tau)f(\xi)\,d\xi. \tag{3b}$$

Die Umkehrung $f = (T^{(\tau)})^{-1}g$ ist jede der beiden Gleichungen

$$\int_0^v f(\xi)\,d\xi = \int_a^\infty d\xi\, g(\xi) \int_0^v \overline{k(x,\xi;\tau)}\,dx = \int_a^\infty d\xi\, g(\xi) \int_0^v k(\xi,x;\bar\tau)\,dx, \tag{3a'}$$

$$f(x) = \underset{n\to\infty}{\mathrm{l.i.m.}} \int_a^n \overline{k(x,\xi;\tau)}g(\xi)\,d\xi = \underset{n\to\infty}{\mathrm{l.i.m.}} \int_a^n k(\xi,x;\bar\tau)g(\xi)\,d\xi. \tag{3b'}$$

* Sei $\mathfrak{L}_2^{(2)}(a,b)$ die Menge aller Funktionen $F(x,y)$, die äquivalent sind messbaren Funktionen $F_1(x,y)$ mit $\|F_1(x,y)\| = \left(\int_a^b\int_a^b |F_1(x,y)|^2\,dxdy\right)^{\frac{1}{2}} < \infty$. Parsevals Satz: Für $f \sim \sum a_n\phi_n$, $h \sim \sum b_n\phi_n$ ist $(f,h) = \sum a_n\bar b_n$.

Beweis. Sei t fest, $|t| < 1$. Aus (3.2) folgt für alle $v \geqslant a$ nach bekannten Sätzen über Konvergenz im Quadratmittel (cf. Wiener, 63, 7.16)

$$\sum_{n=0}^{\infty} t^n \int_0^v \phi_n(y)\, dy\, \bar{\phi}_n(x) \sim \int_0^v k(x,y;t)\, dy. \tag{3.4}$$

Die rechte Seite setze man gleich $H(x,v;t)$. Wegen (3.11) definiert die linke Seite auch für $t = \tau$ ($|\tau| = 1$) eine Funktion $K(x,v;\tau)$, die als Funktion von x zu $\mathfrak{L}_2(a,b)$ gehört. Für $t \to \tau$ strebt nun $H(x,v;t)$ im Quadratmittel bezüglich x gegen $K(x,v;\tau)$. Denn

$$\int_a^{\infty} |K(x,v;\tau)-H(x,v;t)|^2\, dx = \sum |\tau^n - t^n|^2 \left| \int_0^v \phi_n(y)\, dy \right|^2,$$

und die rechte Seite strebt wegen (3.11) für $t \to \tau$ gegen Null (cf. Beweis von I(f)). Für die Folge $\{t_n\}$ nun strebt wegen (iii) nach einem Konvergenzsatze von Lebesgue $H(x,v;t_n)$ für fast alle x gegen $\int_0^v k(x,y;\tau)$, also ist nach einem allgemeinen Satze für alle v und für fast alle x in (a,∞)

$$\int_0^v k(x,y;\tau)\, dy = K(x,v;\tau) \sum_{n=0}^{\infty} \tau^n \bar{\phi}_n(x) \int_0^v \phi_n(y)\, dy. \tag{3.41}$$

Nach Parsevals Satz ist nun für $g = T^{(\tau)} f \sim \sum \tau^n (f,\phi_n)\phi_n$

$$\operatorname{sgn} v\big(g(x), \mathrm{u}(x/v)\big) = \sum_{n=0}^{\infty} \tau^n (f,\phi_n) \int_0^v \phi_n(y)\, dy = \big(f(x), \overline{K(x,v;\tau)}\big) \tag{3.5}$$

mit Rücksicht auf (3.1) und (3.4), also folgt (3a) wegen (3.41).

Sei $f_n(x) = f(x)$ bezw. 0 für $a \leqslant x \leqslant n$ bezw. $x > n$, $g_n = T^{(\tau)} f_n(x)$, d. h.

$$\int_0^v g_n(\xi)\, d\xi = \int_a^n d\xi\, f(\xi) \int_0^v k(\xi,y;\tau)\, dy, \tag{3.51}$$

dann ist $g - g_n = T^{(\tau)}\{f(x) - f_n(x)\}$, und da $T^{(\tau)}$ unitär ist,

$$\|g-g_n\| = \|f(x)-f_n(x)\|,$$

also $\|g-g_n\| \to 0$ für $n \to \infty$. Die Reihenfolge der Integrationen in (3.51) ist beliebig auf Grund absoluter Konvergenz, denn wegen (iii) ist

$$\left(\int_a^n d\xi\, |f(\xi)| \int_0^v |k(\xi,y;\tau)|\, dy\right)^2 \leqslant v \int_a^n |f(\xi)|^2\, d\xi \int_a^n \int_0^v |F(x,y)|^2\, dx dy < \infty.$$

So folgt (3 b) aus $g_n(x) \sim g(x)$ und aus

$$g_n(x) \equiv \frac{d}{dx}\int\limits_0^x dy \int\limits_a^n k(\xi, y; \tau) f(\xi)\, d\xi \equiv \int\limits_a^n k(\xi, x; \tau) f(\xi)\, d\xi.$$

Entsprechend erhält man (3 a′) und (3 b′) aus

$$(T^{(\tau)})^{-1} g = f \sim \sum \bar{\tau}^n (g, \phi_n) \phi_n.$$

Umgekehrt folgt aus (3 b) durch Integrieren und bekannte Sätze über das Quadratmittel (3 a), hieraus (3.5) wegen (3.41) und (3.4), hieraus (3′) mit $t = \tau$ u.s.w.—Man erhält entsprechend den Zusatz

II A. *Im Falle $a = -\infty$ bleibt Satz II gültig mit den Abänderungen: Die in* (iii) *definierte Funktion $F(x, y)$ genüge der Bedingung $F(x, y) \in \mathfrak{L}_2^{(2)}(-A, A)$ für jedes positive A;*

$$g(x) = \underset{n\to\infty}{\text{l.i.m.}} \int\limits_{-n}^{n} k(\xi, x; \tau) f(\xi)\, d\xi, \tag{3′ b}$$

$$f(x) = \underset{n\to\infty}{\text{l.i.m.}} \int\limits_{-n}^{n} k(\xi, x; \bar{\tau}) f(\xi)\, d\xi. \tag{3′ b′}$$

4. Die unitäre Hankelsche Transformation

Sei $\alpha > -1$, $f \in \mathfrak{L}_2(0, \infty)$, dann ist die Transformation

$$g(x) = T_{\frac{1}{2}}^{(\alpha)} f = \underset{n\to\infty}{\text{l.i.m.}} \int\limits_0^n J_\alpha(xy)(xy)^{\frac{1}{2}} f(y)\, dy \tag{4.1}$$

unitär in $\mathfrak{L}_2$, aber für nicht reelles α, $\Re(\alpha) > -1$, nur stetig. Sei

$$\phi_n^{(\alpha)}(x) = \left(\frac{2n!\, e^{-x^2}}{\Gamma(n+\alpha+1)}\right)^{\frac{1}{2}} x^{\alpha+\frac{1}{2}} L_n^{(\alpha)}(x^2); \qquad L_n^{(\alpha)}(z) = \sum_{s=0}^{n} \binom{n+\alpha}{n-s} \frac{(-z)^s}{s!}, \tag{4.2}$$

dann ist $\{\phi_n^{(\alpha)}(x)\}$ $(n = 0, 1, \ldots)$ für $\alpha > -1$ ein vollständiges orthonormales System in $\mathfrak{L}_2$ auf Grund bekannter Eigenschaften der Laguerreschen Polynome. Nun ist für $|t| < 1$*

$$\sum_{n=0}^{\infty} t^n \phi_n^{(\alpha)}(x) \phi_n^{(\alpha)}(y) = \frac{2t^{-\frac{1}{2}\alpha}}{1-t} \exp\left(-\frac{x^2+y^2}{2}\,\frac{1+t}{1-t}\right) I_\alpha\left(\frac{2xy\sqrt{t}}{1-t}\right)(xy)^{\frac{1}{2}}. \tag{4.3}$$

Die durch (3.2) in $\mathfrak{L}_2^{(2)}(0, \infty)$ definierte Funktion $k(x, y; t)$ ist nun für $|t| < 1$ bis auf eine Menge der x, y vom Masze Null gleich der rechten Seite von (4.3); denn $\phi_n^{(\alpha)}$ ist hier reell, $\sum t^n \phi_n^{(\alpha)}(x) \phi_n^{(\alpha)}(y)$ konvergiert

* Formel von Hille und Hardy, cf. G. N. Watson, *J. of London Math. Soc.* 8 (1933), 189–92: $I_\alpha(z) = e^{-\frac{1}{2}i\pi\alpha} J_\alpha(e^{\frac{1}{2}i\pi z})$.

gegen die rechte Seite von (4.3), im Quadratmittel aber gegen $k(x,y;t)$, s. (3.2).

Für jedes t mit $|t| < 1$ gilt folglich (3.3) mit $a = 0$, $b = \infty$.

Sei jetzt $0 < r < 1$, $\tau = e^{2i\pi r}$, $t = R\tau$, $\frac{1}{2} < R < 1$, $R \to 1$, also $t \to \tau$. Dann sind die Bedingungen des Satzes II erfüllt, (ii) wegen

$$k(x,y;t) \to \frac{2e^{-i\pi r\alpha}}{1-e^{2i\pi r}} \exp\left(-i\frac{x^2+y^2}{2}\cot \pi r\right)(xy)^{\frac{1}{2}} I_\alpha\left(\frac{xy}{\sin \pi r} e^{\frac{1}{2}i\pi}\right)$$

$$= k(x,y;\tau) = c_r^{(\alpha)} \exp\left(-i\frac{x^2+y^2}{2}\cot \pi r\right)(xy)^{\frac{1}{2}} J_\alpha\left(\frac{xy}{\sin \pi r}\right), \quad (4.4)$$

$$c_r^{(\alpha)} = \frac{\exp\{i\pi(\frac{1}{2}-r)(1+\alpha)\}}{\sin \pi r};$$

(iii) wegen der unschwer zu erhaltenden Abschätzung

$$|k(x,y;t)| \leqslant F(x,y) = \frac{2^{\frac{1}{2}(1+\alpha)}}{\sin \pi r} I_\alpha\left(\frac{xy}{\sin \pi r}\right)(xy)^{\frac{1}{2}} + \frac{2^{\frac{1}{2}-\alpha}(xy)^{\frac{1}{2}+\alpha}}{\Gamma(\alpha+1)}$$

und wegen $I_\alpha(z) = O(z^\alpha)$ für $z \to 0$, $\frac{1}{2}+\alpha > -\frac{1}{2}$, also

$$F(x,y) \in \mathfrak{L}_2^{(2)}(0,A).$$

Daher ist Satz II anwendbar, und aus (3b) und (4.4) folgt (1), zunächst für $0 < r < 1$, mit Rücksicht auf die Periodizitätseigenschaften $T_{r+1} = T_r$, $((\frac{1}{2}-r-1)) = ((\frac{1}{2}-r))$ u.s.w. für beliebiges reelles, nicht ganzes r. Für ganzes r hat $T_r^{(\alpha)}$ wegen (3) den Sinn $T_0^{(\alpha)} = I$, somit sind für $\alpha > -1$ die Transformationen $T_r^{(\alpha)}$ eine Schar des Satzes I; $\{\phi_n^{(\alpha)}\}$ ist definiert durch (4.2).

Die Ergebnisse gelten auch für $\alpha = -1, -2, \ldots$, denn dann ist

$$F(x,y) = \frac{2^{-\frac{1}{2}(1+\alpha)}}{\sin \pi r} I_\alpha\left(\frac{xy}{\sin \pi r}\right)(xy)^{\frac{1}{2}}; \qquad I_\alpha(z) = I_{-\alpha}(z).$$

Die ersten $-\alpha$ Funktionen der Folge $\phi_n^{(\alpha)}$ sind dann Null.

5. Die Fourier-Transformation

Gehöre $f(x)$ zu $\mathfrak{L}_2(-\infty,\infty)$. Bekanntlich ist $\{\phi_n(x)\}$ $(n = 0, 1, \ldots)$,

$$\phi_n(x) = \frac{e^{-\frac{1}{2}x^2} H_n(x)}{(2^n n! \sqrt{\pi})^{\frac{1}{2}}}; \qquad H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}), \qquad (5.1)$$

ein vollständiges orthonormales System in $\mathfrak{L}_2(-\infty,\infty)$, für $|t| < 1$ ist*

$$\sum_{n=0}^{\infty} t^n \phi_n(x)\phi_n(y) = \{\pi(1-t^2)\}^{-\frac{1}{2}} \exp \frac{4xyt-(x^2+y^2)(1+t^2)}{2(1-t^2)}. \quad (5.2)$$

* Beweis und Quellen: G. N. Watson, *J. of London Math. Soc.* 8 (1933), 194–9; N. Wiener, loc. cit.

Die rechte Seite ist wieder f. ü. gleich $k(x,y;t)$, s. (3.2). Es gilt (3.3) mit $a=-\infty$, $b=\infty$, $|t|<1$. Sei nun $t=R\tau$, $\tau=e^{2i\pi r}$, $\frac{1}{2}<R\to 1$, also $t\to\tau$; $0<r<\frac{1}{2}$ oder $\frac{1}{2}<r<1$. Die Bedingungen des Zusatzes sind erfüllt, (ii) wegen

$$k(x,y;t)\to k(x,y;\tau)=\frac{e^{\frac{1}{2}i\pi((\frac{1}{2}-2r))}}{\sqrt{(2\pi|\sin 2\pi r|)}}\exp\left(\frac{ixy}{\sin 2\pi r}-i\frac{x^2+y^2}{2}\cot 2\pi r\right) \tag{5.3}$$

mit Rücksicht darauf, dass $|\arg\{(1-t^2)^{-\frac{1}{2}}\}|<\frac{1}{4}\pi$ für $|t|<1$ sein muss, (iii) auf Grund der Abschätzung

$$|k(x,y;t)|\leqslant F(x,y)=\frac{1}{\sqrt{(\pi|\sin 2\pi r|)}}\exp\left(\frac{xy}{|\sin 2\pi r|}\right)\in\mathfrak{L}_2^{(2)}(-A,A).$$

Somit folgt (3′ b), hieraus und aus (5.3) die Hauptformel (2) zunächst für $0<r<\frac{1}{2}$ und $\frac{1}{2}<r<1$, auf Grund der Periodizitätseigenschaften $T_{r+1}=T_r$ u.s.w. dann für beliebiges reelles r, wenn $2r$ nicht ganz ist. Im Falle $r\equiv 0 \pmod 1$ ist T_r wieder I, im Falle $r\equiv\frac{1}{2} \pmod 1$ wegen $\phi_n(-x)=(-1)^n\phi_n(x)$

$$\begin{aligned} T_r f(x) &\sim \sum e^{i\pi n}(f,\phi_n)\phi_n(x)\\ &=\sum\big(f(z),\phi_n(-z)\big)\phi_n(x)=\sum\big(f(-z),\phi_n(z)\big)\phi_n(x)\sim f(-x).\end{aligned}$$

Nach Hinzunahme dieser beiden Transformationen bilden also die Transformationen (2) eine Schar des Satzes I; ϕ_n ist bestimmt durch (5.1).

6. Eine weitere Schar unitärer Transformationen

Sei $\mathfrak{H}$ der Raum $\mathfrak{L}_2(0,1)$, $\tau=e^{2i\pi r}$, r reell,

$$\phi_0(x)=1,\qquad \phi_1(x)=\surd 2\cos\pi x,\qquad \phi_2(x)=\surd 2\cos 2\pi x,$$
$$\phi_3(x)=\surd 2\cos 3\pi x,\qquad \ldots,$$

dann ist die Funktion $K(x,y;t)$, für $|t|\leqslant 1$ durch (3.4) u.s.w. bestimmt,

$$K(x,y;t)\sim\sum_{n=0}^{\infty}t^n\phi_n(x)\int_0^y\phi_n(u)\,du\sim h(y+x)+h(y-x);\ |t|\leqslant 1. \tag{6.1}$$

$$h(z;t)=\tfrac{1}{2}z+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{t^n}{n}\sin n\pi z=\tfrac{1}{2}z-\frac{1}{2i\pi}\log\frac{1-te^{i\pi z}}{1-te^{-i\pi z}},$$

wobei $h(z;t)$ eine für alle reellen z und für $|t|<1$, aber auch für $t=\tau=e^{2i\pi r}$ bis auf höchstens $z\equiv\pm 2r \pmod 2$ wohldefinierte Funktion ist. Da aus Stetigkeitsgründen $\mathfrak{I}\log(1-u)$ für $|u|<1$ zwischen

$-\frac{1}{2}\pi$ und $\frac{1}{2}\pi$ zu wählen ist, folgt für $\pm z \not\equiv 2r \pmod 2$ bezw. also $x \not\equiv \pm y \pm 2r \pmod 2$

$$h(z;\tau) = \frac{z}{2} - \frac{1}{2}\left(\left(\frac{1+z}{2}+r\right)\right) + \frac{1}{2}\left(\left(\frac{1-z}{2}+r\right)\right) + \frac{i}{2\pi}\log\left|\frac{\sin\pi(\frac{1}{2}z+r)}{\sin\pi(\frac{1}{2}z-r)}\right|.$$

$$K(x,y;\tau)$$

$$\equiv y - \frac{1}{2}\left(\left(\frac{1+x+y}{2}+r\right)\right) - \frac{1}{2}\left(\left(\frac{1-x+y}{2}+r\right)\right) + \frac{1}{2}\left(\left(\frac{1-x-y}{2}+r\right)\right) +$$

$$+\frac{1}{2}\left(\left(\frac{1+x-y}{2}+r\right)\right) + \frac{i}{2\pi}\log\left|\frac{\cos\pi x - \cos\pi(y+2r)}{\cos\pi x - \cos\pi(y-2r)}\right|.$$

Man erhält schliesslich für $a = 0$, $b = 1$, $f(x) \in L_2(0,1)$ nach einiger Rechnung aus $\big(g(x), \mathfrak{u}(x/z)\big) = \big(f(x), K(x,z;\bar{\tau})\big)$ (s. 3.5)

$$g(z) = T_r f$$

$$\equiv \frac{i}{2\pi}\frac{d}{dz}\int\limits_0^1 \log\left|\frac{\cos\pi x - \cos\pi(z+2r)}{\cos\pi x - \cos\pi(z-2r)}\right| f(x)\,dx + \tfrac{1}{2}f_1(2r+z) + \tfrac{1}{2}f_1(2r-z), \tag{6}$$

wobei $f_1(z)$ diejenige gerade periodische Funktion mit der Periode 2 bedeutet, welche für $0 < z \leqslant 1$ mit $f(z)$ identisch ist. Für $r \equiv \frac{1}{2} \pmod 1$ wird T_r die involutorische Transformation $g(z) \equiv f(1-z)$, für die $\{\phi_{2n}\}$ bezw. $\{\phi_{2n+1}\}$ vollständige orthonormale Systeme der Räume der selbst- bezw. schiefreziproken Funktionen sind. Die Transformationen (6) sind eine Schar des Satzes I.

7. Stetige, aber nicht notwendig unitäre Scharen

Hierzu gehört die Schar (1) für komplexes α ($\Re(\alpha) > -1$); für nicht reelles α ist $\{\phi_n^{(\alpha)}\}$ ($n = 0, 1,\ldots$), s. (4.2), kein orthogonales System.

Sei jetzt $\{\phi_n\}$ ($n = 0, 1,\ldots$) ein in einem Hilbertschen 'Raum $\mathfrak{H}$ abgeschlossenes System',* oder was hier dasselbe sagt, eine Menge, welche 'die abgeschlossene lineare Mannigfaltigkeit $\mathfrak{H}$ bestimmt' ('aufspannt').

Satz III. *Sei das System $\{\phi_n\}$ ($n = 0, 1,\ldots$) in $\mathfrak{H}$ abgeschlossen, η irgend eine Zahl, $0 < |\eta| \leqslant 1$. Dann und nur dann existiert eine lineare stetige Transformation T mit dem Bereiche $\mathfrak{H}$ und der Eigen-*

* Zu jedem $f \in \mathfrak{H}$ gibt es für jedes $\epsilon > 0$ Zahlen $a_0^{(\epsilon)}, a_1^{(\epsilon)},\ldots, a_N^{(\epsilon)}$, $N = N(\epsilon)$ derart, dass $\|f - a_0^{(\epsilon)}\phi_1 - \ldots - a_N^{(\epsilon)}\phi_n\| \leqslant \epsilon$ ist. Die andere Definition: Stone, (1.3), (1.4).

schaft $T\phi_n = \eta^n\phi_n$ $(n = 0,\ 1,\ldots)$, *wenn für jedes Wertesystem* $b_0, b_1,\ldots, b_m$

$$\left\|\sum_{n=0}^{m} b_n\,\eta^n\phi_n\right\| \leqslant A\left\|\sum_{n=0}^{m} b_n\,\phi_n\right\|, \tag{7.1}$$

$A \geqslant 1$ *konstant ist. Es gibt genau ein* T *der verlangten Art.*

Beweis. Da T linear sein soll, folgt $T(\sum b_n\,\phi_n) = \sum b_n\,\eta^n\phi_n$; aus der Stetigkeit, also aus $\|Tf\| \leqslant A\|f\|$ für jedes $f \in \mathfrak{H}$, folgt (7.1) als notwendige Bedingung. Für $b_0 = 1$, $b_1 = b_2 = \ldots = 0$ folgt $A \geqslant 1$.

Die Bedingung ist hinreichend. Sei S die Transformation, deren Bereich ('domain') $\{\phi_n\}$ ist, und für die $S\phi_n = \eta^n\phi_n$ ist, dann existiert genau eine abgeschlossene lineare Erweiterung* ('extension') T von S, und T hat die verlangten Eigenschaften. Sei nämlich die Transformation U festgelegt — eindeutig, da $\sum b_n\,\eta^n\phi_n = \sum b_n'\,\eta^n\phi_n$ wegen (7.1) aus $\sum (b_n - b_n')\phi_n = 0$ folgt — durch

$$U\Big(\sum_{n=0}^{m} b_n\,\phi_n\Big) = \sum_{n=0}^{m} b_n\,\eta^n\phi_n \quad (m,\ b_n \text{ beliebig}),$$

dann ist U die kleinste lineare Erweiterung von S; jede lineare Erweiterung von S ist also auch eine, eigentliche oder uneigentliche, von U; U ist wegen (7.1) stetig, $\|U(X)\| \leqslant A\|X\|$. Nun gilt der leicht beweisbare Satz: Eine lineare stetige Transformation, deren Bereich die abgeschlossene lineare Mannigfaltigkeit $\mathfrak{H}$ bestimmt, besitzt im Hilbertschen Raume $\mathfrak{H}$ genau eine abgeschlossene lineare Erweiterung T; T hat den Bereich $\mathfrak{H}$ und ist stetig, $\|Tf\| \leqslant A\|f\|$ für $f \in \mathfrak{H}$. Wegen $S \subset U \subset T$ und $S\phi_n = \eta^n\phi_n$ ist hier $T\phi_n = \eta^n\phi_n$.

Satz I′. *Sei das System* $\{\phi_n\}$ $(n = 0,\ 1,\ldots)$ *im Hilbertschen Raume* $\mathfrak{H}$ *abgeschlossen, sei bei jedem reellen* r *eine Konstante* A_r *vorhanden derart, dass für* $\eta_r = e^{2i\pi r}$ *und für jedes Zahlensystem* $b_0, b_1,\ldots, b_N$ *stets* $\|\sum b_n\,\eta_r^n\,\phi_n\| \leqslant A_r\,\|\sum b_n\,\phi_n\|$ *ist, sei* T_r *durch Satz III definiert. Die Schar* T_r *hat folgende Eigenschaften:*

(*a*) T_r *ist linear, stetig und hat den Bereich* $\mathfrak{H}$.

(*b*) $T_{r+1} = T_r$, T_0 *ist die identische Transformation* I.

(*c*) (α) $T_s(T_r) = T_{r+s} = T_r(T_s)$, ($\beta$) $T_r^m = T_{mr}$ *für ganzes* m, (γ) $T_r^{-1} = T_{-r}$.

(*d*) *Eigenwerte von* T_r *sind* η_r^k $(k = 0, 1,\ldots)$, *für irrationales* r *sämtlich verschieden, entsprechende Eigenfunktionen sind* ϕ_k.

(*e*) *Sei* r *rational,* $r = n/m$; m, n *ganz und teilerfremd*: (α) *es gibt genau* m *Eigenwerte,* (β) *die zum* k. *Eigenwerte* η_r^k $(0 \leqslant k < m)$ *ge-*

* Hier gebrauchte Begriffe und Sätze: Stone, Def. 2.2, 2.5, Theor. 2.10.

hörigen Eigenfunktionen bilden nach Hinzunahme des Elementes Null einen Hilbertschen Raum $\mathfrak{R}_k$; $\{\phi_{k+sm}\}$ $(s = 0, 1,\ldots)$ *ist ein in* $\mathfrak{R}_k$ *abgeschlossenes System,* (γ) *jede Funktion aus* $\mathfrak{H}$ *ist eindeutig als Summe von höchstens* m *zu verschiedenen Eigenwerten gehörigen Eigenfunktionen darstellbar,* (δ) $T_r^m = T_0 = I$, (ϵ) *wenn* $\{\phi_n\}$ *ein orthogonales System ist, ist* $\mathfrak{R}_k$ *orthogonal zu* $\mathfrak{R}_l$ *für* $k \neq l$.

Zusätze: I′ (f) *Wenn* $A_r \leqslant A$ *für alle* r *ist, ist* $T_s f \sim T_r f$ *für* $s \to r$.

I′ (g). *Wenn ein* T_r *der Schar unitär,* r *irrational ist, ist* $\{\phi_n\}$ *ein vollständiges orthonormales System in* $\mathfrak{H}$ *und jedes* T_r *unitär.*

I′ (h). *Wenn* (7.1) *für ein* η *gilt,* $0 < |\eta| \leqslant 1$, η *keine Einheitswurzel, ist jede endliche Anzahl der* ϕ_n *linear unabhängig.*

Beweis. Die Eigenschaften (a) und (d) folgen aus Satz III; (b) ist einleuchtend. Sei nun

$$\chi = \sum_{n=0}^{N} a_n \phi_n, \qquad \omega = \sum_{n=0}^{N} a_n \eta_r^n \phi_n = T_r \chi, \qquad \zeta = \sum_{n=0}^{N} (a_n \eta_r^n) \eta_s^n \phi_n = T_s \omega.$$

Wegen $\eta_r \eta_s = e^{2i\pi(r+s)} = \eta_{r+s}$ ist $T_s(T_r \chi) = T_{r+s} \chi$; da man jedes f aus $\mathfrak{H}$ durch Funktionen χ im Quadratmittel approximieren kann und T_r, T_s, T_{r+s} stetig sind, folgt $(c(\alpha))$, hieraus $(c(\gamma))$ wegen

$$T_r T_{-r} = T_{-r} T_r = T_0 = I,$$

ferner $(c(\beta))$, für negative m mit Rücksicht auf $(c(\gamma))$, hieraus $(e(\delta))$. Für $r = n/m$ sind $\eta_r^0, \eta_r^1,\ldots, \eta_r^{m-1}$ voneinander verschieden, $\eta_r^l = \eta_r^k$ für $l \equiv k \pmod{m}$. Aber eine lineare Transformation, die einer Beziehung der Art $(e(\delta))$ genügt und mindestens m verschiedene Eigenwerte hat, ist von ‘algebraischem Typ mit dem Grade m’. Eine solche* Transformation hat genau m verschiedene Eigenwerte, jede Funktion ihres Bereiches $\mathfrak{D}_T$ ist eindeutig als Summe von höchstens m zu verschiedenen Eigenwerten gehörigen Eigenfunktionen darstellbar; wenn die Transformation abgeschlossen ist, ist $\mathfrak{R}_k$ $(k = 0, 1,\ldots, m-1)$, die Menge der zum k. Eigenwerte gehörigen Eigenfunktionen unter Hinzunahme des Elementes Null, eine abgeschlossene lineare Mannigfaltigkeit; wenn sie stetig ist und $\{\psi_{k,l}\}$ $(k = 0, 1,\ldots, m-1;$ $l = 0, 1,\ldots)$ ein in $\mathfrak{D}_T$ abgeschlossenes System, und wenn jede Funktion $\psi_{k,l}$ zu $\mathfrak{R}_k$ gehört $(l = 0, 1,\ldots)$, ist $\{\psi_{k,l}\}$ (k fest; $l = 0, 1,\ldots$) ein in $\mathfrak{R}_k$ abgeschlossenes System.

* Definition: T ist linear, für jedes $f \in \mathfrak{D}$ erfüllt Tf die Gleichung

$$T^m + a_1 T^{m-1} + \ldots + a_m T^0 = 0; \qquad P_m(x) = x^m + a_1 x^{m-1} + \ldots + a_m \quad (m \geqslant 1)$$

hat nur einfache Nullstellen; es gibt kein $P_n(x)$, $n < m$ mit $P_n(Tf) = 0$ für alle $f \in \mathfrak{D}$: Verf., Def. B, Satz I bis V, erscheint in *Annals of Math.*

Hier ist $\mathfrak{D}_T = \mathfrak{H}$, T_r abgeschlossen, $\psi_{k,l} = \phi_{k+lm}$; $\mathfrak{R}_k$ enthält beliebig viele linear unabhängige Elemente ϕ_{k+lm}, ist also ein Hilbertscher Raum; so folgt $(e(\alpha))$, $(e(\beta))$ und $(e(\gamma))$.—Der Beweis für $(e(\epsilon))$ und die Zusätze wird nicht ausgeführt.

8. Anwendung auf die Hankel-Transformation

Hilfssatz. *Sei α komplex, $\Re(\alpha) > 1$, $\phi_n^{(\alpha)}$ durch* (4.2) *bestimmt. Die Transformation* (1) *$g = T_r^{(\alpha)} f$ existiert und ist stetig, $\|T_r^{(\alpha)} f\| \leqslant A\|f\|$, A eine nur von α abhängige Konstante; ferner ist $T_r^{(\alpha)}\phi_n^{(\alpha)} = e^{2i\pi rn}\phi_n^{(\alpha)}$.*

Das System $\{\phi_n^{(\alpha)}\}$ ist bekanntlich in $\mathfrak{L}_2(0,\infty)$ abgeschlossen, $T_r^{(\alpha)}$ ersichtlich linear, somit folgt aus dem Hilfssatz, dass $T_r^{(\alpha)}$ mit der Transformation des Satzes III für $\eta = e^{2i\pi r}$, $\mathfrak{H} = \mathfrak{L}_2(0,\infty)$ identisch ist; also gilt (7.1), natürlich auch für ganzes r, somit hat die Schar $T_r^{(\alpha)}$, bestimmt durch (1) unter Hinzunahme von $T_0^{(\alpha)} = I$ u.s.w., sämtliche in Satz I′ ausgesprochenen Eigenschaften.

Zusatz.* *Ist r irrational, so gibt es ausser $e^{2i\pi rn}$ bezw. $a\phi_n^{(\alpha)}$ $(n = 0, 1, \ldots)$ keine weiteren Eigenwerte bezw. Eigenfunktionen.*

9. Beweis des Hilfssatzes

Bekanntlich ist

$$\sqrt{z}\,J_\alpha(z) = \beta\cos z + \gamma\sin z + L(z),$$

$$|L(z)| < Az^{\frac{1}{2}+\Re(\alpha)} \quad (0 < z \leqslant 1), \qquad |L(z)| < Az^{-1} \quad (z > 1). \tag{9.1}$$

Sei $f(x) \in \mathfrak{L}_2(0,\infty)$, $0 < a \leqslant b$; $f(x; a, b)$ gleich $f(x)$ für $a \leqslant x \leqslant b$, sonst Null; $|\sin \pi r| = \rho$,

$$g_a(x) = c_r^{(\alpha)} \int\limits_0^a J_\alpha\!\left(\frac{xy}{\rho}\right)(xy)^{\frac{1}{2}} e^{-\frac{1}{2}i(x^2+y^2)\cot\pi r} f(y)\,dy, \tag{9.2}$$

r nicht ganz, $\Re(\alpha) > -1$, das Integral daher konvergent. So ist

$$\frac{g_b(x)-g_a(x)}{c_r^{(\alpha)}\sqrt{\rho}} = \left(\int\limits_0^\infty \beta\cos\frac{xy}{\rho} + \int\limits_0^\infty \gamma\sin\frac{xy}{\rho}\right) e^{-\frac{1}{2}i(x^2+y^2)\cot\pi r} f(y; a, b)\,dy +$$

$$+\left(\int\limits_0^{\rho/x} + \int\limits_{\rho/x}^\infty\right) L\!\left(\frac{xy}{\rho}\right) e^{-\frac{1}{2}i(x^2+y^2)\cot\pi r} f(y; a, b)\,dy = I_1 + I_2 + I_3 + I_4.$$

* Diese Frage bleibt im allgemeinen Falle noch offen.

Die Transformation (1) ist in den Fällen $\alpha = \mp\frac{1}{2}$ sicher unitär, so folgt

$$\|I_1(x)\| = |\beta|(\pi\rho/2)^{\frac{1}{2}}\|f(y;a,b)\|, \qquad \|I_2(x)\| = |\gamma|(\pi\rho/2)^{\frac{1}{2}}\|f(y;a,b)\|.$$

Aus (9.1) folgt

$$|I_3(\rho x)| \leqslant A\int_0^{1/x} (xy)^{\frac{1}{2}+\Re(\alpha)}|f(y;a,b)|\,dy,$$

$$|I_4(\rho x)| \leqslant A\int_{1/x}^{\infty} (xy)^{-1}|f(y;a,b)|\,dy,$$

hieraus für $k = 3, 4$: $\|I_k(\rho x)\| \leqslant A\|f(y;a,b)\|$,*

$$\|I_k(x)\| = \sqrt{\rho}\,\|I_k(\rho x)\| \leqslant A\sqrt{\rho}\|f(y;a,b)\| \quad (\rho = |\sin\pi r|;\ k = 3, 4),$$

schliesslich

$$\|g_b(x)-g_a(x)\| \leqslant |c_r^{(\alpha)}\sin\pi r|A'\|f(y;a,b)\| \leqslant A\|f(y;a,b)\| \tag{9.3}$$

$$= A\Big(\int_a^b |f(y)|^2\,dy\Big)^{\frac{1}{2}} \to 0 \text{ für } b > a \to \infty.$$

Somit konvergiert $g_a(x)$ im Quadratmittel gegen eine Funktion $g(x)$ aus $\mathfrak{L}_2(0,\infty)$, aus (9.2) folgt (1). Diese Transformation existiert also und ist gleichmässig für alle zulässigen r beschränkt; denn aus (9.3) folgt für $a \to 0$, $b \to \infty$ leicht $\|g(x)\| = \|T_r^{(\alpha)}f\| \leqslant A\|f\|$.

Nach Satz I ist $T_r^{(\alpha)}\phi_n^{(\alpha)} = e^{2i\pi nr}\phi_n^{(\alpha)}$ für $\alpha > -1$ (s. Abschn. 4),

$$\text{d. h. } c_r^{(\alpha)}\int_0^\infty J_\alpha\Big(\frac{xy}{|\sin\pi r|}\Big)(xy)^{\frac{1}{2}}e^{-\frac{1}{2}i(x^2+y^2)\cot\pi r}\phi_n^{(\alpha)}(y)\,dy = e^{2i\pi nr}\phi_n^{(\alpha)}(x), \tag{9.4}$$

wobei das Integral zunächst im Quadratmittel, ersichtlich aber auch absolut konvergiert. Aus Stetigkeitsgründen gilt (9.4) für *alle* $x > 0$; beide Seiten sind für $\alpha_0 \leqslant \Re(\alpha) \leqslant \alpha_1$, $|\Im(\alpha)| \leqslant A$, wenn

$$-1 < \alpha_0 < \alpha_1 < \infty$$

ist, auf Grund gleichmässiger Konvergenz des Integrales reguläre Funktionen von α. Daher gilt (9.4) und folglich $T_r^{(\alpha)}\phi_n^{(\alpha)} = e^{2i\pi nr}\phi_n^{(\alpha)}$ für $\Re(\alpha) > -1$. Der Hilfssatz ist somit bewiesen.

* Verf., *Quart. J. of Math.* (Oxford), 8 (1937), 186–99, Hilfssatz 4 B, 4 C, $\eta = -\frac{1}{2}-\Re(\alpha)$, $\rho = 1$, $p = q = 2$. Zu diesem Abschnitte: cf. E. C. Titchmarsh, *J. of London Math. Soc.* 1 (1926), 195–6.

10. Beweis des Zusatzes

Bekanntlich ist

$$(\phi_m^{(\alpha)}, \overline{\phi_n^{(\alpha)}}) = 0 \text{ bezw. } 1 \quad \text{für } m \neq n \text{ bezw. } m = n. \tag{10.1}$$

Sei ein von allen $\eta_r^n = e^{2i\pi rn}$ $(n = 0, 1,...)$ verschiedener Eigenwert ξ vorhanden, f eine zugehörige Eigenfunktion. Dann müssen zu jedem $\epsilon > 0$ Zahlen $a_n^{(\epsilon)}$ $(n = 0, 1,...)$; $a_n^{(\epsilon)} = 0$ für $n > N(\epsilon)$, vorhanden sein, derart, dass

$$\|f - \sum a_n^{(\epsilon)} \phi_n^{(\alpha)}\| \leqslant \tfrac{1}{2}\epsilon/\max(A, |\xi|). \tag{10.2}$$

Folglich wäre $\|T_r^{(\alpha)}f - \sum a_n^{(\epsilon)} \eta_r^n \phi_n^{(\alpha)}\| \leqslant \frac{1}{2}\epsilon$, also wegen $T_r^{(\alpha)}f = \xi f$

$$\sum_{n=0}^{\infty} = \sum_{n=0}^{N(\epsilon)} a_n^{(\epsilon)}(\xi - \eta_r^n)\phi_n^{(\alpha)} = F, \qquad \|F\| \leqslant \epsilon. \tag{10.3}$$

Durch Multiplizieren mit $\phi_n^{(\alpha)}$ und Integrieren folgt wegen (10.1) für jedes n

$$|a_n^{(\epsilon)}(\xi - \eta_r^n)| = |(F, \overline{\phi_n^{(\alpha)}})| \leqslant \epsilon\|\phi_n^{(\alpha)}\| \quad (n = 0, 1,...), \tag{10.4}$$

also $a_n^{(\epsilon)} \to 0$ für $\epsilon \to 0$. Ebenso folgt aus (10.2) wegen $A \geqslant 1$

$$|(f, \overline{\phi_n^{(\alpha)}}) - a_n^{(\epsilon)}| \leqslant \tfrac{1}{2}\epsilon\|\phi_n^{(\alpha)}\| \to 0 \quad \text{für } \epsilon \to 0, \tag{10.5}$$

also $(f, \overline{\phi_n^{(\alpha)}}) = 0$, wegen der Vollständigkeit von $\{\overline{\phi_n^{(\alpha)}}\}$ in $\mathfrak{L}_2$ daher $f \equiv 0$; ξ kann somit kein Eigenwert sein.

Sei jetzt $\xi = \eta_r^k = e^{2i\pi rk}$, r irrational, dann folgt aus (10.4) $a_n^{(\epsilon)} \to 0$ für $n \neq k$, aus (10.5) also $(f, \overline{\phi_n^{(\alpha)}}) = 0$ für $n \neq k$. Sei $(f, \overline{\phi_k^{(\alpha)}}) = a$. Auf Grund von (10.1) erhält man $(f - a\phi_k^{(\alpha)}, \overline{\phi_n^{(\alpha)}}) = 0$ $(n = 0, 1,...)$; daher ist $f - a\phi_k^{(\alpha)} = 0$, $a\phi_k^{(\alpha)}$ die einzige zu η_r^k gehörige Eigenfunktion.

Die Schar (1) besitzt also für nicht reelles α alle Eigenschaften, welche denen des unitären Falles entsprechen.

ON THE PRODUCT OF TWO LAGUERRE POLYNOMIALS

By W. N. BAILEY (*Manchester*)

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1. In recent years several formulae have been given which express the product of two Laguerre polynomials as a sum involving these polynomials. A formula of my own* is concerned with polynomials with different arguments, but with the same order and exponent. This is

$$L_n^{(\alpha)}(x)L_n^{(\alpha)}(y) = \frac{\Gamma(1+\alpha+n)}{n!}\sum_{r=0}^{n}\frac{(xy)^r L_{n-r}^{(\alpha+2r)}(x+y)}{r!\,\Gamma(1+\alpha+r)}. \tag{1.1}$$

A formula involving two Laguerre polynomials with the same argument and exponent but with different orders, obtained independently by Howell† and Erdélyi,‡ is

$$L_m^{(\alpha)}(z)L_n^{(\alpha)}(z) = \frac{\Gamma(1+\alpha+m)\Gamma(1+\alpha+n)}{\Gamma(1+\alpha+m+n)}\times$$

$$\times\sum_{r=0}^{\min(m,n)}\frac{(m+n-2r)!}{r!\,(m-r)!\,(n-r)!\,\Gamma(1+\alpha+r)}z^{2r}L_{m+n-2r}^{(\alpha+2r)}(z). \tag{1.2}$$

Another result for the square of a Laguerre polynomial, due to Howell,§ is

$$[L_n^{(\alpha)}(z)]^2 = \frac{\Gamma(1+\alpha+n)}{2^{2n}n!}\sum_{r=0}^{n}\frac{(2r)!\,(2n-2r)!}{r!\,\{(n-r)!\}^2\Gamma(1+\alpha+r)}L_{2r}^{(2\alpha)}(2z). \tag{1.3}$$

Watson|| has obtained a formula of the type

$$L_m^{(\alpha)}(z)L_n^{(\alpha)}(z) = \sum_{r=0}^{2\min(m,n)} c_r\, L_{m+n-r}^{(\alpha)}(z) \tag{1.4}$$

in which the coefficients c_r are given in terms of a series ${}_3F_2(1)$, and

* W. N. Bailey, *Proc. London Math. Soc.* (2), 41 (1936), 215–20 (5.4).

† W. T. Howell, *Phil. Mag.* (7), 24 (1937), 396–405 (29).

‡ A. Erdélyi, *Monatshefte für Math. und Phys.* 46 (1937), 132 (4.5).

§ W. T. Howell, *Phil. Mag.* (7), 24 (1937), 1082–93 (7.1). I have corrected a slip in Howell's result.

|| G. N. Watson, *J. of London Math. Soc.* 13 (1938), 29–32.

Erdélyi* has shown how an equivalent formula can be obtained from (1.2) and his relation

$$z^m L_n^{(\alpha+m)}(z) = \frac{\Gamma(1+\alpha+m+n)}{n!} \sum_{r=0}^{m} \frac{(-m)_r (n+r)!}{r!\,\Gamma(1+\alpha+n+r)} L_{n+r}^{(\alpha)}(z). \quad (1.5)$$

The more general problem of obtaining the expansion

$$L_{m_1}^{(\alpha_1)}(k_1 z) \ldots L_{m_n}^{(\alpha_n)}(k_n z) = \sum_{r=0}^{m_1+\ldots+m_n} c_r\, L_r^{(\alpha)}(z) \quad (1.6)$$

has been completely solved, in one sense, by Erdélyi,† who showed that the coefficient c_r can be expressed either as a Lauricella's hypergeometric function F_A of $n+1$ variables, or as an n-ple sum. In the case of a product of two polynomials, the coefficient c_r is thus expressed as a double series, but it is not at all easy to express it as a simple sum. The formula (1.3), for example, appears to be difficult to obtain from Erdélyi's general result.

I give here some further expansions for the product of two Laguerre polynomials in which the exponents are different, the coefficients being expressed in comparatively simple forms.

2. I first prove the formula

$$L_n^{(\alpha)}(z) L_n^{(\beta)}(z)$$
$$= \frac{\Gamma(1+\alpha+n)\Gamma(1+\beta+n)}{n!} \sum_{r=0}^{n} \frac{(-1)^r z^r L_r^{(\alpha+\beta+r)}(z)}{\Gamma(1+\alpha+r)\Gamma(1+\beta+r)(n-r)!} \quad (2.1)$$

in which the polynomials have different exponents but equal orders. In the proof we need the transformation‡ of a nearly-poised series of the type ${}_3F_2(-1)$, namely

$${}_3F_2\left[\begin{matrix}-n, & b, & d; \\ & 1-n-b, & w\end{matrix}\; -1\right]$$

$$= \frac{(w-d)_n}{(w)_n}\, {}_4F_3\left[\begin{matrix}d, & -\tfrac{1}{2}n, & \tfrac{1}{2}(1-n), & 1-n-w; \\ & 1-n-b, & \tfrac{1}{2}(1+d-w-n), & 1+\tfrac{1}{2}(d-w-n)\end{matrix}\right]. \quad (2.2)$$

* A. Erdélyi, *J. of London Math. Soc.* 13 (1938), 154–6.

† See the last paper cited.

‡ See W. N. Bailey, *Generalized Hypergeometric Series* (Cambridge tract, 1935), § 4.7, where $c \to \infty$.

We thus have

$$L_n^{(\alpha)}(z)L_n^{(\beta)}(z)$$
$$= \frac{(1+\alpha)_n(1+\beta)_n}{(n!)^2}\, {}_1F_1(-n, 1+\alpha; z)\, {}_1F_1(-n, 1+\beta; z)$$
$$= \frac{(1+\alpha)_n(1+\beta)_n}{(n!)^2} \sum_{N=0}^{2n} \frac{(-n)_N z^N}{N!\,(1+\beta)_N}\, {}_3F_2\left[\begin{matrix}-N, & -n, & -\beta-N; \\ & 1+n-N, & 1+\alpha\end{matrix}\ -1\right]$$
$$= \frac{(1+\alpha)_n(1+\beta)_n}{(n!)^2} \sum_{N=0}^{2n} \frac{(-n)_N(1+\alpha+\beta+N)_N z^N}{N!\,(1+\beta)_N(1+\alpha)_N} \times$$
$$\times\, {}_4F_3\left[\begin{matrix}-\beta-N, & -\tfrac{1}{2}N, & \tfrac{1}{2}(1-N), & -\alpha-N; \\ & 1-N+n, & \tfrac{1}{2}(-\alpha-\beta-2N), & \tfrac{1}{2}(1-\alpha-\beta-2N)\end{matrix}\right]$$
$$= \frac{(1+\alpha)_n(1+\beta)_n}{(n!)^2} \times$$
$$\times \sum_{N=0}^{2n} \sum_{r=0}^{[\frac{1}{2}N]} \frac{(-n)_N(1+\alpha+\beta+N)_N(-\alpha-N)_r(-\beta-N)_r(-N)_{2r} z^N}{N!\,(1+\alpha)_N(1+\beta)_N\, r!\,(1-N+n)_r(-\alpha-\beta-2N)_{2r}}$$
$$= \frac{(1+\alpha)_n(1+\beta)_n}{(n!)^2} \sum_{N=0}^{2n} \sum_{r=0}^{[\frac{1}{2}N]} \frac{(-1)^r(-n)_{N-r}(1+\alpha+\beta)_{2N-2r} z^N}{(N-2r)!\,(1+\alpha)_{N-r}(1+\beta)_{N-r}(1+\alpha+\beta)_N\, r!}.$$

Putting $N = r+s$, we get

$$L_n^{(\alpha)}(z)L_n^{(\beta)}(z)$$
$$= \frac{(1+\alpha)_n(1+\beta)_n}{(n!)^2} \sum_{s=0}^{n} \sum_{r=0}^{s} \frac{(-1)^r(-n)_s(1+\alpha+\beta)_{2s} z^{r+s}}{r!\,(s-r)!\,(1+\alpha)_s(1+\beta)_s(1+\alpha+\beta)_{r+s}}$$
$$= \frac{(1+\alpha)_n(1+\beta)_n}{(n!)^2} \sum_{s=0}^{n} \frac{(-n)_s(1+\alpha+\beta)_{2s} z^s}{s!\,(1+\alpha)_s(1+\beta)_s(1+\alpha+\beta)_s}\, {}_1F_1[-s; 1+\alpha+\beta+s; z],$$

and this proves the result.

3. We now prove the formula

$$z^n L_n^{(\alpha+n)}(z) = \frac{\Gamma(1+\alpha+2n)}{2^{2n}} \sum_{r=0}^{n} \frac{(-1)^r(2r)!}{r!\,(n-r)!\,\Gamma(1+\alpha+2r)} L_{2r}^{(\alpha)}(2z), \quad (3.1)$$

which will be needed to transform (2.1). To prove this, we need the known formula*

$$z^n = \Gamma(1+\alpha+n) \sum_{r=0}^{n} \frac{(-n)_r}{\Gamma(1+\alpha+r)} L_r^{(\alpha)}(z) \quad (3.2)$$

* See, for example, A. Erdélyi, *Math. Zeitschrift*, 42 (1937), 641 (2.6).

with z replaced by $2z$. Then

$$z^n L_n^{(\alpha+n)}(z)$$

$$= \frac{(1+\alpha+n)_n}{n!} \sum_{r=0}^{n} \frac{(-n)_r}{r!\,(1+\alpha+n)_r} z^{n+r}$$

$$= \frac{(1+\alpha+n)_n}{n!} \sum_{r=0}^{n} \sum_{s=0}^{n+r} \frac{(-n)_r}{r!\,(1+\alpha+n)_r} \frac{\Gamma(1+\alpha+n+r)(-n-r)_s}{2^{n+r}\Gamma(1+\alpha+s)} L_s^{(\alpha)}(2z)$$

$$= \frac{\Gamma(1+\alpha+2n)}{2^n} \sum_{s=0}^{2n} \frac{(-1)^s L_s^{(\alpha)}(2z)}{\Gamma(1+\alpha+s)(n-s)!} {}_2F_1\left[\begin{matrix} -n, & n+1; \\ & n-s+1 \end{matrix} \; \tfrac{1}{2}\right].$$

The series ${}_2F_1$ can be summed by the formula*

$${}_2F_1\left[\begin{matrix} a, & 1-a; \\ & c \end{matrix} \; \tfrac{1}{2}\right] = \frac{\Gamma(\frac{1}{2}c)\Gamma(\frac{1}{2}c+\frac{1}{2})}{\Gamma(\frac{1}{2}c+\frac{1}{2}a)\Gamma(\frac{1}{2}+\frac{1}{2}c-\frac{1}{2}a)},$$

a limiting form of the result being used when $s > n$, and (3.1) is immediately obtained.

4. We are now in a position to transform (2.1). Using (3.1), we find that

$$L_n^{(\alpha)}(z) L_n^{(\beta)}(z) = \frac{(1+\alpha)_n (1+\beta)_n}{n!} \times$$

$$\times \sum_{r=0}^{n} \sum_{p=0}^{r} \frac{(-1)^r \Gamma(1+\alpha+\beta+2r)}{2^{2r}(1+\alpha)_r(1+\beta)_r(n-r)!} \frac{(-1)^p (2p)!\, L_{2p}^{(\alpha+\beta)}(2z)}{p!\,(r-p)!\,\Gamma(1+\alpha+\beta+2p)}.$$

Putting $r = p+s$, we find that

$$L_n^{(\alpha)}(z) L_n^{(\beta)}(z) = \frac{(1+\alpha)_n (1+\beta)_n}{n!} \sum_{p=0}^{n} c_p \, L_{2p}^{(\alpha+\beta)}(2z), \tag{4.1}$$

where

$$c_p = \frac{(\frac{1}{2})_p}{(1+\alpha)_p(1+\beta)_p(n-p)!} \times$$

$$\times {}_3F_2\left[\begin{matrix} \frac{1}{2}(1+\alpha+\beta+2p), & \frac{1}{2}(2+\alpha+\beta+2p), & -n+p; \\ 1+\alpha+p, & 1+\beta+p & \end{matrix}\right].$$

The coefficient c_p is thus expressed as a simple sum. When $\beta = \alpha$, the ${}_3F_2$ reduces to a ${}_2F_1$ which can be summed, and we obtain (1.3).

* W. N. Bailey, *Generalized Hypergeometric Series* (Cambridge tract, 1935), § 2.4 (3).

Another case in which the coefficients simplify is when $\beta = \alpha+1$, and then we obtain the formula

$$L_n^{(\alpha)}(z)L_n^{(\alpha+1)}(z) = \frac{\Gamma(1+\alpha+n)}{2^{2n}n!}\sum_{r=0}^{n}\frac{(2r)!\,(2n-2r)!}{r!\,\{(n-r)!\}^2\Gamma(1+\alpha+r)}L_{2r}^{(2\alpha+1)}(2z). \tag{4.2}$$

From the recurrence formula

$$L_n^{(\alpha)}(z) = L_n^{(\alpha+1)}(z) - L_{n-1}^{(\alpha+1)}(z)$$

and (1.3) and (4.2) we obtain

$$L_n^{(\alpha)}(z)L_{n-1}^{(\alpha+1)}(z) = \frac{\Gamma(1+\alpha+n)}{2^{2n}n!}\sum_{r=1}^{n}\frac{(2r)!\,(2n-2r)!}{r!\,\{(n-r)!\}^2\Gamma(1+\alpha+r)}L_{2r-1}^{(2\alpha+1)}(2z). \tag{4.3}$$

In particular, when $\alpha = -\frac{1}{2}$, we have from (4.2) and (4.3) the expansions

$$H_{2n}(x)H_{2n+1}(x) = 2^{2n+1}(2n)!\,x\sum_{r=0}^{n}\frac{(2r)!}{2^{2r}(r!)^2}L_{2n-2r}(2x^2), \tag{4.4}$$

$$H_{2n-1}(x)H_{2n}(x) = -2^{2n}(2n-1)!\,x\sum_{r=0}^{n-1}\frac{(2r)!}{2^{2r}(r!)^2}L_{2n-2r-1}(2x^2), \tag{4.5}$$

where $H_n(x)$ is Hermite's polynomial.

The last two results can be combined into the single formula

$$H_n(x)H_{n+1}(x) = (-1)^n 2^{n+1}n!\,x\sum_{r=0}^{[\frac{1}{2}n]}\frac{(2r)!}{2^{2r}(r!)^2}L_{n-2r}(2x^2). \tag{4.6}$$

When written in terms of parabolic cylinder functions of integral order, (4.6) becomes

$$e^{\frac{1}{2}x^2}D_n(x)D_{n+1}(x) = (-1)^n n!\,x\sum_{r=0}^{[\frac{1}{2}n]}\frac{(2r)!}{2^{2r}(r!)^2}L_{n-2r}(x^2). \tag{4.6 a}$$

Howell* has pointed out that Watson's integral equation† for the square of a Laguerre polynomial

$$e^{-x}x^{\alpha}\{L_n^{(\alpha)}(x)\}^2 = \int_0^\infty J_{2\alpha}\{2\sqrt{(xy)}\}e^{-y}y^{\alpha}\{L_n^{(\alpha)}(y)\}^2\,dy \tag{4.7}$$

follows from (1.3) by mere summation, by using the simple result

$$\int_0^\infty J_\alpha\{2\sqrt{(xs)}\}e^{-s}s^{\frac{1}{2}\alpha}L_n^{(\alpha)}(2s)\,ds = (-1)^n e^{-x}x^{\frac{1}{2}\alpha}L_n^{(\alpha)}(2x). \tag{4.8}$$

* See Howell's second paper, cited above.

† G. N. Watson, *J. of London Math. Soc.* 11 (1936), 256–61.

It will be noticed that Erdélyi's generalization of Watson's result,* namely,

$$e^{-x}x^{\frac{1}{2}(\alpha+\beta)}L_n^{(\alpha)}(x)L_n^{(\beta)}(x) = \int_0^\infty J_{\alpha+\beta}\{2\sqrt{(xy)}\}e^{-y}y^{\frac{1}{2}(\alpha+\beta)}L_n^{(\alpha)}(y)L_n^{(\beta)}(y)\,dy \tag{4.9}$$

follows in the same way from (4.1), since the right-hand side of (4.1) contains only polynomials of even order.

Erdélyi's integral equations†

$$\int_0^\infty J_0(xy)D_{2n-1}(y)D_{2n}(y)\,dy = -x^{-1}D_{2n-1}(x)D_{2n}(x),$$

$$\int_0^\infty J_0(xy)D_{2n}(y)D_{2n+1}(y)\,dy = x^{-1}D_{2n}(x)D_{2n+1}(x)$$

follow in the same way from (4.6 a).

5. I take this opportunity of noting that a simple proof of (1.2) can be obtained by combining the formula

$$\sum_{n=0}^\infty \frac{t^n L_n^{(\alpha)}(z)}{\Gamma(1+\alpha+n)} = e^t(tz)^{-\frac{1}{2}\alpha}J_\alpha\{2\sqrt{(tz)}\} \tag{5.1}$$

with the expansion‡

$$J_\nu(z\cos\theta)J_\nu(z\sin\theta) = \sum_{r=0}^\infty \frac{(\frac{1}{4}z\sin 2\theta)^{\nu+2r}}{r!\,\Gamma(\nu+r+1)}J_{\nu+2r}(z). \tag{5.2}$$

For

$$\sum_{m=0}^\infty\sum_{n=0}^\infty \frac{t^m u^n L_m^{(\alpha)}(z)L_n^{(\alpha)}(z)}{\Gamma(1+\alpha+m)\Gamma(1+\alpha+n)} = e^{t+u}(tuz^2)^{-\frac{1}{2}\alpha}J_\alpha\{2\sqrt{(tz)}\}J_\alpha\{2\sqrt{(uz)}\}$$

$$= e^{t+u}(tuz^2)^{-\frac{1}{2}\alpha}\sum_{r=0}^\infty \frac{1}{r!\,\Gamma(1+\alpha+r)}\left(\frac{tuz}{t+u}\right)^{\frac{1}{2}\alpha+r} J_{\alpha+2r}[2\sqrt{\{z(t+u)\}}]$$

$$= \sum_{r=0}^\infty\sum_{s=0}^\infty \frac{(z^2tu)^r}{r!\,\Gamma(1+\alpha+r)}\frac{(t+u)^s L_s^{(\alpha+2r)}(z)}{\Gamma(1+\alpha+s+2r)},$$

and (1.2) is obtained by equating the coefficients of $t^m u^n$ on the two sides.

* A. Erdélyi, ibid. 13 (1938), 146–54.

† A. Erdélyi, loc. cit. (3.14) and (3.15).

‡ See my paper cited in § 1, (5.2).

6. The following simple and elegant proof of (2.1) is due to Mr. Chaundy. By using the formula

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha}}{n!}(D-1)^n x^{\alpha+n},$$

where $D \equiv d/dx$, (2.1) becomes

$$(D-1)^n x^{\alpha+n}(D-1)^n x^{\beta+n}$$
$$= \sum_{r=0}^{n} (-1)^r \binom{n}{r} (\alpha+n)_{-(n-r)}(\beta+n)_{-(n-r)}(D-1)^r x^{\alpha+\beta+2r}, \quad (6.1)$$

where $$(a)_{-n} \equiv a(a-1)(a-2)...(a-n+1).$$

Now, if D_1, $D_2 \equiv d/dx_1$, d/dx_2, we have

$$(D_1-1)^n x_1^{\alpha+n}(D_2-1)^n x_2^{\beta+n} = \{(D_1-1)(D_2-1)\}^n x_1^{\alpha+n} x_2^{\beta+n}$$
$$= \{D_1 D_2-(D_1+D_2-1)\}^n x_1^{\alpha+n} x_2^{\beta+n}$$
$$= \sum_{r=0}^{n} (-1)^n \binom{n}{r} (D_1+D_2-1)^r (D_1 D_2)^{n-r} x_1^{\alpha+n} x_2^{\beta+n}$$
$$= \sum_{r=0}^{n} (-1)^n \binom{n}{r} (\alpha+n)_{-(n-r)}(\beta+n)_{-(n-r)}(D_1+D_2-1)^r x_1^{\alpha+r} x_2^{\beta+r}.$$

We now identify x_1, x_2 with x, and then D_1+D_2 is equivalent to D, and (6.1) is proved.

NOTE ON DIFFERENTIATED FOURIER SERIES

By L. S. BOSANQUET (*London*)

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1. Introduction. Sufficient conditions for the summability of the series $\sum n(b_n \cos nx - a_n \sin nx)$, obtained by differentiating the terms of a Fourier series, are well known. A typical result is that *the differentiated series is summable* (C, α) $(\alpha > 1)$ *to the derivative of the function at all points at which the function is differentiable, the function being assumed integrable L.** A sufficient condition for the summability $(C, \alpha+1)$ $(\alpha = 0, 1, 2, ...)$ of the differentiated Fourier series of $f(t)$ at $t = x$ was given by Rajchman† and by W. H. Young,‡ namely, that§ $\psi(t)/(2\tan\frac{1}{2}t) = \{f(x+t) - f(x-t)\}/(4\tan\frac{1}{2}t)$ *should be integrable L and its Fourier series be summable* (C, α) *at* $t = x$. Necessary conditions have also been given.‖ The nearest approach to a condition which is both necessary and sufficient was made by Zygmund, who obtained the following result:**

THEOREM A. *If $f(t)$ is integrable L in $(-\pi, \pi)$, and if the function $\chi(t) = \psi(t)/(2\tan\frac{1}{2}t)$ is integrable L, then a necessary and sufficient condition for the differentiated Fourier series of $f(t)$ to be summable $(C, \alpha+1)$ for $t = x$ to sum s, where $\alpha \geqslant 0$, is that the Fourier series of $\chi(t)$ be summable (C, α) to s for $t = 0$.*

Zygmund's theorem, however, does not give us a general necessary condition for the differentiated series to be summable (C, k). In fact there are differentiated series which are summable (C), for which the function $\psi(t)/t$ is not absolutely integrable. For example, if $x = 0$ and $f(t) = \psi(t) = \sin(1/t)$, the differentiated Fourier series, as we shall show later, is summable (C, k) $(k > 3)$, but $|\sin(1/t)|/t$ has not a finite integral over any range including the origin.

The first object of this note is to show that the hypothesis that $\psi(t)/t$ should be integrable L may be removed, and that $\psi(t)/t$ must, in fact, possess an integral of a slightly more general type, whenever

* See A. Zygmund, *Trigonometrical Series*, p. 55.

† A. Rajchman, *Prace Mat. Fiz.* 28 (1917), 213–19.

‡ W. H. Young, *Proc. London Math. Soc.* (2) 17 (1918), 195–236.

§ We write $\psi(t) = \frac{1}{2}\{f(x+t) - f(x-t)\}$ throughout the paper. Young employed the function $\psi(2t)/(2\sin t)$ instead of $\psi(t)/(2\tan\frac{1}{2}t)$.

‖ See Zygmund, loc. cit., p. 260.

** A. Zygmund, *Studia Math.* 3 (1931), 77–91.

the differentiated series is summable (C). We then obtain a complete solution of the Cesàro summability problem for the derived series, in the sense that we reduce it to the problem of the summability of the Fourier series of $\psi(t)/(2\sin\frac{1}{2}t)$,* the coefficients in the latter series being defined by means of the generalized integral.

By starting with a function which is integrable L we try to avoid unnecessary generality, but we are forced by the analysis to introduce a definition of integration, which we term Cesàro-Lebesgue and which is, in fact, included in the Cesàro-Perron definition, recently introduced by Burkill.† It is not, however, necessary for our purpose to have any knowledge of the Perron integral. We therefore begin by defining independently the least general integral that we shall require.

2. The Cesàro-Lebesgue integral. Suppose that $\phi(t)$ is integrable L in the interval (ϵ, a), for every ϵ such that $0 < \epsilon < a$, and suppose that λ is a non-negative integer. *If there is a function* $\Phi_{\lambda+1}(t)$ *such that* (i) $(d/dt)^{\lambda+1}\Phi_{\lambda+1}(t) = \phi(t)$ *for almost all* t *in* $(0, a)$, (ii) $\Phi_{\lambda+1}(t) = o(t^{\lambda})$ *as* $t \to +0$, *we say that* $\phi(t)$ *is integrable* $C_\lambda L$, *i.e. in the Cesàro-Lebesgue sense of order* λ, *in* $(0, a)$.‡

Plainly, if $\phi(t)$ is integrable $C_\lambda L$, then it is also integrable $C_{\lambda+1} L$, the function $\Phi_{\lambda+2}(t)$ being the integral of $\Phi_{\lambda+1}(u)$ over the range $(0, t)$. Further, if $\Phi_s(t) = (d/dt)^{\lambda+1-s}\Phi_{\lambda+1}(t)$, for $s = 1, 2, \ldots, \lambda$, then $\Phi_s(t)$ is integrable $C_{\lambda-s} L$.

If $\phi(t)$ is integrable $C_\lambda L$ and $\Phi_{\lambda+1}(t) \sim st^{\lambda+1+p}/\Gamma(\lambda+p+2)$ as $t \to +0$, where $p > -1$, we write $\phi(t) \sim st^p$ $(C, \lambda+1)$ as $t \to +0$; then the related notation, such as $\phi(t) = o(t^p)$ $(C, \lambda+1)$, $\phi(t) \to s$ $(C, \lambda+1)$, is interpreted in the obvious way.

Plainly, if $\phi(t)$ is integrable $C_\lambda L$, we have

$$\Phi_s(t) = o(t^{s-1}) \quad (C, \lambda-s+1);$$

in particular $\Phi_1(t) = o(1)$ (C, λ).

* It is immaterial whether we argue with $\psi(t)/(2\tan\frac{1}{2}t)$, $\psi(t)/(2\sin\frac{1}{2}t)$, or $\psi(t)/t$, provided that the last two are defined by periodicity outside $(-\pi, \pi)$, for the difference between any pair of them satisfies Dini's convergence criterion at the origin, whenever $\psi(t)$ is integrable L.

† J. C. Burkill, *Proc. London Math. Soc.* (2) 34 (1932), 314–22, and (2) 39 (1935), 541–52. Ours is the case when there is just one point of non-absolute integrability in the range of periodicity.

‡ The definition may be extended to cover cases when there are other points of non-absolute integrability or when λ is not an integer, but we shall not require these in this paper.

We now define the $C_\lambda L$ integral of $\phi(t)$ over $(0,a)$ by the equation

$$\int\limits_{\to 0(C,\lambda)}^{a} \phi(u)\,du = (C,\lambda)\text{-}\lim_{\epsilon\to+0} \int\limits_{\epsilon}^{a} \phi(u)\,du. \tag{1}$$

Plainly the $C_0 L$ integral is an ordinary Cauchy-Lebesgue integral.

We now give some properties of the Cesàro-Lebesgue integral which we shall require.

LEMMA 1. *If $p > -1$, $r < p+1$, $\phi(t)$ is integrable $C_\lambda L$ in $(0,a)$ and $\phi(t) = o(t^p)$ $(C,\lambda+1)$ as $t \to +0$, and if $\theta(t) = O(t^{-r})$ and $\theta^{(\rho)}(t) = O(t^{-r-\rho})$ in $(0,a)$, for $\rho = 1, 2,\dots, \lambda$, then $\phi(t)\theta(t)$ is integrable $C_\lambda L$ in $(0,a)$ and is $o(t^{p-r})$ $(C,\lambda+1)$ as $t \to +0$.*

If $0 < \epsilon < t \leqslant a$, we have

$$\int\limits_{\epsilon}^{t} \phi(u)\theta(u)\,du = [\Phi_1(u)\theta(u)]_\epsilon^t - \int\limits_{\epsilon}^{t} \Phi_1(u)\theta'(u)\,du. \tag{2}$$

First suppose $\lambda = 0$. Then $\Phi_1(t) = o(t^{p+1})$, and it follows at once that the right-hand side of (2) is $o(t^{p+1-r})+o(\epsilon^{p+1-r})$. Hence, since $p+1-r > 0$, $\phi(u)\theta(u)$ is integrable $C_0 L$ and is $o(t^{p-r})$ $(C,1)$.

Next suppose $\lambda = s$, where s is a positive integer, and suppose that the lemma is true for $\lambda = s-1$. Then, since $\phi(t) = o(t^p)$ $(C,s+1)$, we have $\Phi_1(t) = o(t^{p+1})$ (C,s). It follows from the case $\lambda = s-1$, with $\Phi_1(t)$ in place of $\phi(t)$, and $p+1$ in place of p, that $\Phi_1(t)\theta(t)$ is integrable $C_{s-1} L$ and is $o(t^{p+1-r})$ (C,s) as $t \to +0$. Also, by the case $\lambda = s-1$, with $\Phi_1(t)$ in place of $\phi(t)$, $p+1$ in place of p, $r+1$ in place of r, and $\theta'(t)$ in place of $\theta(t)$, $\Phi_1(t)\theta'(t)$ is integrable $C_{s-1} L$ and is $o(t^{p-r})$ (C,s) as $t \to +0$. Since $p+1-r > 0$, it follows that the right-hand side of (2) tends to a limit (C,s) as $\epsilon \to 0$, and that

$$\int\limits_{\to 0(C,s)}^{t} \phi(u)\theta(u)\,du = o(t^{p+1-r}) \quad (C,s)$$

as $t \to +0$, i.e. $\phi(t)\theta(t) = o(t^{p-r})$ $(C,s+1)$.

The result now follows by induction.

LEMMA 2. *If $\phi(t)$ is integrable $C_\lambda L$ in $(0,a)$, then $\phi(t)e^{i\mu t}$ is integrable $C_\lambda L$, and*

$$\int\limits_{\to 0(C,\lambda)}^{a} \phi(t)\,e^{i\mu t}\,dt = \Phi_1(a)\,e^{i\mu a} - i\mu \int\limits_{\to 0(C,\lambda-1)}^{a} \Phi_1(t)\,e^{i\mu t}\,dt. \tag{3}$$

If $0 < \epsilon < a$, then

$$\int_{\epsilon}^{a} \phi(t)\, e^{i\mu t}\, dt = [\Phi_1(t)\, e^{i\mu t}]_{\epsilon}^{a} - i\mu \int_{\epsilon}^{a} \Phi_1(t)\, e^{i\mu t}\, dt. \tag{4}$$

Now $\Phi_1(t)$ is integrable $C_{\lambda-1} L$ and is $o(1)$ as $t \to +0$. Therefore, by Lemma 1, with $\theta(t) = e^{i\mu t}$ and $p = r = 0$, $\Phi_1(t)\, e^{i\mu t}$ is integrable $C^{*}_{\lambda-1} L$ and is $o(1)$ (C, λ) as $t \to +0$. The result follows immediately.

We end this section by stating without proof that, *if $\phi(t)$ is integrable $C_\lambda L$ in $(0, a)$, then it is also integrable $C_\lambda P$.**

3. The main theorem. In this section we obtain the following result:

THEOREM 1. *If $\alpha \geqslant 0$, and if $f(t)$ is integrable L in $(-\pi, \pi)$, and of period 2π, then a necessary and sufficient condition that the differentiated Fourier series of $f(t)$ should be summable $(C, \alpha+1)$ to s at the point $t = x$ is that the even function $\psi(t)/(2\sin\frac{1}{2}t)$ should be integrable in the Cesàro-Lebesgue sense in $(0, \pi)$ and its Fourier series*† *be summable (C, α) to s at $t = 0$.*

We write $$f(t) \sim \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt),$$

so that the differentiated series is

$$\sum_{n=1}^{\infty} n(b_n \cos nt - a_n \sin nt) = \sum_{n=1}^{\infty} nB_n(t),$$

and $$\psi(t) \sim \sum_{n=1}^{\infty} B_n \sin nt,$$

where $B_n = B_n(x)$.

LEMMA 3. *If $f(t)$ is integrable L in $(-\pi, \pi)$ and $\sum nB_n$ is summable (C) to s, then $\psi(t) \sim st$ (C) as $t \to +0$.*

If we replace $\psi(t)$ by $\chi(t) = \psi(t) - s\sin t$, we replace s by 0. We may therefore suppose, without loss of generality, that $\sum nB_n$ is summable (C, k) to zero, where k is an integer. Writing S_n^{μ} for the

* For the definition of integrability $C_\lambda P$ see Burkill, loc. cit.

† i.e. $\frac{1}{2}\alpha(0) + \sum_{n=1}^{\infty} \alpha(n) \cos nt$, where $\alpha(n) = \dfrac{2}{\pi} \displaystyle\int_{\to 0(C)}^{\pi} \frac{\psi(t)}{2\sin\frac{1}{2}t} \cos nt\, dt.$

nth Cesàro sum of order μ of the series $\sum nB_n$, we have*

$$\frac{(k+1)!\,\Psi_{k+1}(t)}{t^{k+1}}$$

$$= (k+1)\int_0^1 (1-u)^k\psi(tu)\,du = (k+1)\sum_{n=1}^{\infty} B_n\,\bar{\gamma}_{k+1}(nt)$$

$$= t\sum_{n=1}^{\infty} nB_n\,\gamma_{k+2}(nt) = t\sum_{n=1}^{\infty} S_n^k\,\Delta^{k+1}\gamma_{k+2}(nt)$$

$$= t\Big\{\sum_{n\leqslant t^{-1}} + \sum_{n>t^{-1}}\Big\} = t\{\textstyle\sum_1 + \sum_2\}.$$

Then, as $t \to +0$,

$$\textstyle\sum_1 = \sum_{n\leqslant t^{-1}} o(n^k)\,O(t^{k+1}) = o(1)$$

and
$$\textstyle\sum_2 = \sum_{n>t^{-1}} o(n^k)\,O(n^{-k-2}t^{-1}) = o(1),$$

and hence $\Psi_{k+1}(t) = o(t^{k+2})$.

LEMMA 4. *If $\psi(t)$ is integrable L in $(0,\pi)$, and if $\psi(t) \sim st$ (C) as $t \to +0$, then $\psi(t)/(2\sin\frac{1}{2}t)$ is integrable in the Cesàro-Lebesgue sense in $(0,\pi)$ and tends to s (C) as $t \to +0$.*

This follows from Lemma 1 with $\phi(t) = \psi(t)-st$ and

$$\theta(t) = (2\sin\tfrac{1}{2}t)^{-1}.$$

LEMMA 5. *If $\phi(t)$ is even and integrable in the Cesàro-Lebesgue sense in $(0,\pi)$, and if $\phi(t) \to s$ (C) as $t \to +0$, then the Fourier series of $\phi(t)$ is summable (C) to s for $t = 0$.*

This was proved by Burkill† for a function integrable in the Cesàro-Perron sense. By using Lemma 2 we may adapt Burkill's analysis to the present case.

* Cf. L. S. Bosanquet, *Proc. London Math. Soc.* (2) 31 (1930), 135–143. We write, for $\sigma > 0$,

$$\gamma_\sigma(x)+i\bar{\gamma}_\sigma(x) = \int_0^1 (1-u)^{\sigma-1}e^{ixu}\,du.$$

Then
$$\sigma x^{-1}\bar{\gamma}_\sigma(x) = \gamma_{\sigma+1}(x)$$

and
$$|\Delta^\mu\gamma_\sigma(nt)| \leqslant \frac{At^\mu}{1+(nt)^\sigma+(nt)^{\mu+2}} \quad (t \geqslant 0;\ n \geqslant 0;\ \mu = 1, 2,\ldots),$$

where A is independent of n and t, and

$$\Delta f(n) = f(n)-f(n+1), \qquad \Delta^\mu = \Delta(\Delta^{\mu-1}), \qquad \Delta^1 = \Delta.$$

The partial summations are easily justified, since $B_n = o(1)$.

† J. C. Burkill, *J. of London Math. Soc.* 10 (1935), 254–9.

LEMMA 6. *If $\phi(t)$ is integrable in the Cesàro-Lebesgue sense in $(0,\pi)$ and $t\phi(t)$ is integrable L, and if*

$$\alpha(\mu) = \frac{2}{\pi} \int_{\to 0(C)}^{\pi} \phi(t)\cos \mu t\, dt, \tag{5}$$

then, if one of the series $\frac{1}{2}\alpha(0)+\sum_{n=1}^{\infty}\alpha(n)$ and $\sum_{n=0}^{\infty}\alpha(n+\frac{1}{2})$ is summable (C,α) with sum s, where $\alpha \geqslant 0$, so is the other.

Observing that

$$\sigma_n = \tfrac{1}{2}\alpha(0)+\sum_{\nu=1}^{n}\alpha(\nu) = \frac{2}{\pi}\int_{\to 0(C)}^{\pi} \phi(t)\frac{\sin(n+\frac{1}{2})t}{2\sin\frac{1}{2}t}\,dt \tag{6}$$

and

$$\tau_n = \sum_{\nu=0}^{n}\alpha(\nu+\tfrac{1}{2}) = \frac{2}{\pi}\int_{\to 0(C)}^{\pi} \phi(t)\frac{\sin(n+1)t}{2\sin\frac{1}{2}t}\,dt, \tag{7}$$

we easily verify that

$$2\tau_{n-1}-\sigma_n-\sigma_{n-1} = \frac{2}{\pi}\int_{0}^{\pi} \phi(t)\tan\tfrac{1}{4}t\sin nt\,dt = o(1) \tag{8}$$

as $n\to\infty$, by the Riemann-Lebesgue theorem, since $\phi(t)\tan\frac{1}{4}t$ is integrable L. By a similar argument we see that

$$2\sigma_n-\tau_n-\tau_{n-1} = o(1). \tag{9}$$

The result follows from (8) and (9).

LEMMA 7.* *If $\sum_{n=0}^{\infty} u_n$ is summable (C), then a necessary and sufficient condition that $\sum_{n=1}^{\infty} n(u_{n-1}-u_n)$ should be summable $(C,\alpha+1)$ to s, where $\alpha > -1$, is that $\sum u_n$ should be summable (C,α) to s.*

This follows from the identity†

$$t_n^{\alpha+1} = \frac{n}{n+\alpha+1}s_{n-1}^{\alpha+1}-(\alpha+1)(s_n^{\alpha}-s_n^{\alpha+1}) \tag{10}$$

and the consistency theorem for Cesàro limits, where t_n^{σ} and s_n^{σ} are the nth Cesàro means of order σ of $\sum n\,\Delta u_{n-1}$ and $\sum u_n$ ($t_0^0 = 0$).

* This is included in a more general result of G. H. Hardy and J. E. Littlewood, *Math. Zeits.* 19 (1924), 67–96. The sufficiency part, for $\alpha = 0,1,2,\ldots$, was given by Rajchman, loc. cit., and (essentially) by E. Fekete, *Math. és Term. Ért.* 32 (1914), 389–425.

† Cf. C. E. Winn, *J. of London Math. Soc.* 7 (1932), 227–30.

Proof of Theorem 1. First suppose that $\sum nB_n$ is summable $(C, \alpha+1)$ to s. It follows, by Lemma 3, that $\psi(t) \sim st\ (C)$ as $t \to +0$, and hence, by Lemma 4, that $\psi(t)/(2\sin\frac{1}{2}t)$ is integrable in the Cesàro-Lebesgue sense and tends to $s\ (C)$ as $t \to +0$. Hence, by Lemma 5, its Fourier series, $\frac{1}{2}\alpha(0)+\sum\alpha(n)\cos nt$, is summable (C) for $t = 0$, and hence, by Lemma 6, $\sum\alpha(n+\frac{1}{2})$ is summable (C). Now

$$\Delta\alpha(n-\tfrac{1}{2}) = \frac{2}{\pi}\int\limits_{\to 0(C)}^{\pi} \frac{\psi(t)}{2\sin\frac{1}{2}t}\Delta\cos(n-\tfrac{1}{2})t = \frac{2}{\pi}\int\limits_{0}^{\pi}\psi(t)\sin nt\,dt = B_n.$$

It follows, by the necessity part of Lemma 7, that $\sum\alpha(n+\frac{1}{2})$ is summable (C,α) to s, and hence, by Lemma 6, $\frac{1}{2}\alpha(0)+\sum\alpha(n)$ is also summable (C,α) to s.

Conversely, if $\psi(t)/(2\sin\frac{1}{2}t)$ is integrable in the Cesàro-Lebesgue sense, and its Fourier series, $\frac{1}{2}\alpha(0)+\sum\alpha(n)\cos nt$, is summable (C,α) to s at $t = 0$, then, by Lemma 6, $\sum\alpha(n+\frac{1}{2})$ is summable (C,α) to s, and hence, by the sufficiency part of Lemma 7, since $\Delta\alpha(n-\frac{1}{2}) = B_n$, $\sum nB_n$ is summable $(C,\alpha+1)$ to s.

4. Example. We have already observed that $\{\sin(1/t)\}/t$ is not absolutely integrable. An easy calculation shows that

$$\sin(1/t) = o(t) \quad (C, 2),$$

and hence, by Lemma 1, $\{\sin(1/t)\}/(2\sin\frac{1}{2}t)$ is integrable $C_1 L$ and is $o(1)\ (C,2)$ as $t \to +0$. It follows, by the proof of Lemma 5, that its Fourier series is summable $(C, 2+\delta)$ $(\delta > 0)$ at $t = 0$, and hence, by Theorem 1, the differentiated Fourier series of $\sin(1/t)$ is summable $(C, 3+\delta)$ at $t = 0$.

5. It is perhaps worth remarking that Theorem 1 may be obtained more quickly by quoting some known results. If $\sum nB_n$ is summable (C), then $\sum B_n$ must be summable (C),* and hence

$$\int\limits_{t}^{\pi}\frac{\psi(u)}{2\sin\frac{1}{2}u}\,du,$$

which is integrable L in $(0,\pi)$ whenever $\psi(t)$ is integrable L,† tends to a limit (C) as $t \to +0$,‡ i.e. $\psi(t)/(2\sin\frac{1}{2}t)$ is integrable in the

* H. Bohr, *Comptes rendus*, 148 (1909), 75–80.

† G. H. Hardy, *Messenger of Math.* 58 (1928), 50–2.

‡ G. H. Hardy and J. E. Littlewood, *Proc. London Math. Soc.* (2) 24 (1925), 211–46.

Cesàro-Lebesgue sense. Thus Lemmas 3 and 4 may be avoided. On the other hand, the method we have chosen may be generalized, and we can prove the following theorem.

THEOREM 2. *If $\alpha \geqslant 0$ and $f(t)$ is integrable L, then a necessary and sufficient condition that the r-times differentiated Fourier series of $f(t)$ should be summable $(C, \alpha+r)$ to s at $t = x$ is that there should exist an algebraic polynomial $P(t)$, of degree $(r-1)$, such that*

$$\frac{\{f(x+t)+P(t)\}+(-1)^r\{f(x-t)+P(-t)\}}{2(2\sin\frac{1}{2}t)^r/(r!)} \tag{11}$$

*is integrable in the Cesàro-Lebesgue sense and its Fourier series is summable (C, α) to s at $t = 0$.**

* The sufficiency of the condition when the function (11) is integrable L was shown (essentially) by W. H. Young, loc. cit., for $\alpha = 0, 1, 2,\ldots$, and by F. T. Wang, *Tôhoku Math. Journ.* 39 (1934), 399–405, for $\alpha > 0$. $P(t)$ may also be replaced by a trigonometrical polynomial.

[*Added 30 Dec. 1938.* The statement at the end of § 2 is, in the case $\lambda = 1$, a corollary of a theorem of Miss M. E. Grimshaw. *Proc. Cambridge Phil. Soc.* 30 (1934), 15–18. The general case is the same in principle.]

AN EXAMPLE IN THE THEORY OF THE SPECTRUM OF A FUNCTION

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THE 'spectrum' of the function

$$f(x) = \sum_{n=1}^{N} a_n e^{i\lambda_n x} \quad (\lambda_n \text{ real})$$

may be defined to be a non-decreasing step-function $S(u)$ with a step of magnitude $|a_n|^2$ at $u = \lambda_n$. Thus

$$S(b)-S(a) = \sum_{a<\lambda_n<b} |a_n|^2,$$

a factor $\frac{1}{2}$ being inserted in a term where $\lambda_n = a$ or b.

The idea was generalized by N. Wiener,* who proved that, if $f(x)$ is any function such that

$$\phi(x) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x+t)\bar{f}(t)\, dt$$

exists for all real x, then

$$S(u) = \frac{1}{2\pi} \lim_{X\to\infty} \int_{-X}^{X} \phi(x) \frac{e^{ixu}-1}{ix}\, dx$$

exists for all real u, and is a bounded non-decreasing function of u. This is called by Wiener the spectrum of $f(x)$, and the definition is consistent with the previous one.

Wiener considers the further generalization in which, instead of a single $f(x)$, we are given an aggregate of functions, and the problem is to determine the probable value of $S(u)$ for a function selected at random from the aggregate. Examples of this are given by Wiener and Kenrick.† Kenrick considers functions which take the values 1 and -1 alternately, the changes in value occurring at random, with an average frequency of occurrence of k changes per unit length. The apparatus required for dealing rigorously with such situations seems to be rather complicated, and it is not easy to see in a 'given' case exactly what aggregate of functions one is supposed to be considering. For this reason I work out the corresponding example in

* N. Wiener, *Acta Math.* 55 (1930), 117–258, and *The Fourier Integral*, Cambridge, 1933.

† G. W. Kenrick, *Phil. Mag.* (7), 7 (1929), 176–96.

which the changes of sign occur at the points q/p (p a large positive integer, $q = 0, \pm 1, \pm 2,...$) only. The aggregate of functions defined over a finite interval is finite, so that the calculations are quite elementary; and it will be seen that the resulting formulae all tend to limits as $p \to \infty$.

For a finite aggregate of functions, the probability of a given relation being true is the proportion of functions for which it is true. Consider a finite aggregate of functions $f(x)$, each defined over $-T_1 \leqslant x \leqslant T_1$, where T_1 is a large positive even integer. Instead of $\phi(x)$, we consider

$$\phi(x, T) = \frac{1}{2T} \int_{-T}^{T} f(x+t)\bar{f}(t)\, dt.$$

This is defined for $|x|+T \leqslant T_1$. Take $T = \frac{1}{2}T_1$, so that $\phi(x, T)$ is defined for $|x| \leqslant T$. We have then to consider the probability that

$$S(u, T) = \frac{1}{2\pi} \int_{-T}^{T} \phi(x, T)\frac{e^{ixu}-1}{ix}\, dx$$

lies within certain limits for large values of T. If all the functions of the aggregate are the same, the problem reduces to the previous one, viz. the determination of $S(u)$.

Consider first functions $f(x)$ defined over $(0, 1)$, starting with a given $f(0)$, and taking the values 1 and -1 only. Imagine the function splitting into $a+b$ branches at $x = 0$, a with a change of sign and b without. Let each branch split into $a+b$ branches at $x = 1/p$, a with a change of sign and b without; and so on up to $(p-1)/p$. The total number of cases is $(a+b)^p$, and to each corresponds a definite function $f(x)$. The functions $f(x)$ are, however, not necessarily all different.

The total number of changes of sign is

$$a(a+b)^{p-1}+a(a+b)^{p-1}+\ldots \text{ to } p \text{ terms} = pa(a+b)^{p-1}.$$

Hence the average number of changes per function is

$$\frac{pa(a+b)^{p-1}}{(a+b)^p} = \frac{pa}{a+b}.$$

Given a rational k, we choose a and b so that this is equal to k; if k is an integer we may, for example, take $a = k$, $b = p-k$. Then the average number of changes of sign per unit length is k. The process may obviously be extended to any interval. To any finite

interval corresponds a finite number of functions $f(x)$. The number of functions defined in the interval $(-T_1, T_1)$ is $(a+b)^{2pT_1}$.

Let $g_n(q/p)$ be the probability that $f(x)$ has exactly n changes of sign in the interval $t \leqslant x < t+q/p$. This is independent of t, so that we may take $t = 0$. Then it is clear from the definitions that

$$g_0\left(\frac{1}{p}\right) = \frac{b}{a+b} = 1-\frac{k}{p}, \qquad g_0\left(\frac{q+1}{p}\right) = \left(1-\frac{k}{p}\right)g_0\left(\frac{q}{p}\right),$$

so that
$$g_0\left(\frac{q}{p}\right) = \left(1-\frac{k}{p}\right)^q.$$

For $n \geqslant 1$, the probability that there are n changes of sign in $(0, (q+1)/p)$ is equal to the probability that there are $n-1$ changes of sign in $(0, q/p)$, multiplied by the probability of a change at q/p; plus the probability of n changes in $(0, q/p)$, multiplied by the probability of no change at q/p;

i.e.
$$g_n\left(\frac{q+1}{p}\right) = \frac{k}{p}g_{n-1}\left(\frac{q}{p}\right)+\left(1-\frac{k}{p}\right)g_n\left(\frac{q}{p}\right).$$

Also
$$g_1\left(\frac{1}{p}\right) = \frac{k}{p}, \qquad g_2\left(\frac{1}{p}\right) = 0,\ldots.$$

It is easily verified that the solution of these equations is

$$g_n\left(\frac{q}{p}\right) = \begin{cases} \dfrac{q(q-1)\ldots(q-n+1)}{n!}\left(\dfrac{k}{p}\right)^n\left(1-\dfrac{k}{p}\right)^{q-n} & (n \leqslant q), \\ 0 & (n > q). \end{cases}$$

Now for any t the value of $f(q/p+t)f(t)$ is 1 or -1 according to whether an even or odd number of changes of sign occur in $(t, t+q/p)$. Thus, if $q > 0$,

$$\begin{aligned} \text{average}\, f\left(\frac{q}{p}+t\right)f(t) &= g_0\left(\frac{q}{p}\right)-g_1\left(\frac{q}{p}\right)+g_2\left(\frac{q}{p}\right)-\ldots \\ &= \left\{\left(1-\frac{k}{p}\right)-\frac{k}{p}\right\}^q = \left(1-\frac{2k}{p}\right)^q. \end{aligned}$$

Let $\bar{\phi}(x, T)$ denote the average of $\phi(x, T)$ over all functions of the aggregate. Then

$$\bar{\phi}\left(\frac{q}{p}, T\right) = \left(1-\frac{2k}{p}\right)^q \quad \left(0 < \frac{q}{p} \leqslant T\right).$$

If $f(x) = c_n$ in the interval $(n/p, (n+1)/p)$, then

$$\phi\left(\frac{q}{p}, T\right) = \frac{1}{2pT}(c_{q-pT}\,c_{-pT}+\ldots+c_{q+pT-1}\,c_{pT-1}),$$

while, if $q/p < x < (q+1)/p$,

$$\phi(x,T) = \frac{1}{2T}\left\{\int\limits_{-T}^{-T+\frac{q+1}{p}-x} + \int\limits_{-T+\frac{q+1}{p}-x}^{-T+\frac{1}{p}} + \ldots + \int\limits_{T-\frac{1}{p}}^{T-\frac{1}{p}+\frac{q+1}{p}-x} + \int\limits_{T-\frac{1}{p}+\frac{q+1}{p}-x}^{T}\right\} f(x+t)f(t)\,dt$$

$$= \frac{(q+1)/p-x}{2T}(c_{q-pT}\,c_{-pT}+\ldots)+\frac{x-q/p}{2T}(c_{q-pT+1}\,c_{-pT}+\ldots).$$

Hence $$\phi(x,T) = (q+1-px)\phi\left(\frac{q}{p},T\right)+(px-q)\phi\left(\frac{q+1}{p},T\right),$$

i.e. $\phi(x,T)$ is linear in the intervals between the points q/p. Hence $\tilde{\phi}(x,T)$ is also linear. Further, it is clear that $\phi(x,T)$ and $\tilde{\phi}(x,T)$ are even functions of x.

If we define $\tilde{\phi}(x)$ for all x as being even, 1 for $x=0$ and $(1-2k/p)^q$ for $x=q/p$, and linear in the intervals between these points, then

$$\tilde{\phi}(x,T) = \tilde{\phi}(x) \qquad (|x| \leqslant T).$$

If $\tilde{S}(u,T)$ is the average of the $S(u,T)$,

$$\tilde{S}(u,T) = \frac{1}{2\pi}\int\limits_{-T}^{T}\tilde{\phi}(x,T)\frac{e^{ixu}-1}{ix}\,dx = \frac{1}{\pi}\int\limits_{0}^{T}\tilde{\phi}(x)\frac{\sin xu}{x}\,dx.$$

As $T\to\infty$, $\tilde{S}(u,T)$ tends to the limit

$$\tilde{S}(u) = \frac{1}{\pi}\int\limits_{0}^{\infty}\tilde{\phi}(x)\frac{\sin xu}{x}\,dx.$$

Now $$\frac{d}{du}\tilde{S}(u) = \frac{1}{\pi}\int\limits_{0}^{\infty}\tilde{\phi}(x)\cos xu\,dx$$

$$= \frac{1}{\pi}\sum_{q=0}^{\infty}\left\{\left[\tilde{\phi}(x)\frac{\sin xu}{u}\right]_{q/p}^{(q+1)/p} - \frac{1}{u}\int\limits_{q/p}^{(q+1)/p}\tilde{\phi}'(x)\sin xu\,dx\right\}$$

$$= \frac{1}{\pi u^2}\sum_{q=0}^{\infty}p\left\{\left(1-\frac{2k}{p}\right)^q-\left(1-\frac{2k}{p}\right)^{q+1}\right\}\left\{\cos\frac{qu}{p}-\cos\frac{(q+1)u}{p}\right\}$$

$$= \frac{2k(1-k/p)\sin^2\frac{1}{2}u/p}{\pi u^2\{(1-2k/p)\sin^2\frac{1}{2}u/p+k^2/p^2\}}.$$

Hence

$$\tilde{S}(u) = \frac{2k(1-k/p)}{\pi}\int\limits_0^u \frac{\sin^2\frac{1}{2}v/p}{(1-2k/p)\sin^2\frac{1}{2}v/p+k^2/p^2}\,\frac{dv}{v^2}.$$

If we make p and q tend to infinity in such a way that $q/p \to x$, we obtain

$$\lim g_n\left(\frac{q}{p}\right) = \frac{x^n k^n}{n!}\,e^{-kx}, \qquad \lim \tilde{\phi}(x) = e^{-2kx},$$

and

$$\lim \tilde{S}(u) = \frac{2k}{\pi}\int\limits_0^u \frac{dv}{v^2+4k^2} = \frac{1}{\pi}\arctan\frac{u}{2k}.$$

These are the formulae given by Kenrick.

The example given by Wiener, in which $f(x)$ is equally likely to be 1 or -1 in each of the intervals $(q, q+1)$, is got by putting $p = 1$, $k = \frac{1}{2}$.

We can now calculate the probability that $\phi(x, T)$ will differ from $\tilde{\phi}(x)$, or $S(u, T)$ from $\tilde{S}(u)$, by not more than a given number δ. By enumerating the cases in which $c_\mu c_{\mu+q} c_\nu c_{\nu+q}$ is 1 or -1, it is easily seen that, if $q > 0$,

$$\operatorname{av} c_\mu c_{\mu+q} c_\nu c_{\nu+q} = \left\{g_0\left(\frac{q}{p}\right) - g_1\left(\frac{q}{p}\right) + \dots\right\}^2 = \left(1-\frac{2k}{p}\right)^{2q}.$$

Hence

$$\begin{aligned}
\operatorname{av}\left\{\phi\left(\frac{q}{p}, T\right)\right\}^2 &= \frac{1}{4p^2T^2}\operatorname{av}\sum_{\mu=-pT}^{pT-1}\sum_{\nu=-pT}^{pT-1} c_\mu c_{\mu+q} c_\nu c_{\nu+q} \\
&= \frac{1}{4p^2T^2}\left\{\sum_\mu \operatorname{av} c_\mu^2 c_{\mu+q}^2 + \sum_{\mu\neq\nu}\sum \operatorname{av} c_\mu c_{\mu+q} c_\nu c_{\nu+q}\right\} \\
&= \frac{1}{4p^2T^2}\left\{2pT + (4p^2T^2 - 2pT)\left(1-\frac{2k}{p}\right)^{2q}\right\} \\
&= \left(1-\frac{2k}{p}\right)^{2q} + \frac{1}{2pT}\left\{1-\left(1-\frac{2k}{p}\right)^{2q}\right\}.
\end{aligned}$$

Hence

$$\begin{aligned}
\operatorname{av}\left\{\phi\left(\frac{q}{p}, T\right) - \tilde{\phi}\left(\frac{q}{p}\right)\right\}^2 &= \operatorname{av}\left\{\phi\left(\frac{q}{p}, T\right)\right\}^2 - \left\{\tilde{\phi}\left(\frac{q}{p}\right)\right\}^2 \\
&= \frac{1}{2pT}\left\{1-\left(1-\frac{2k}{p}\right)^{2q}\right\}.
\end{aligned}$$

Hence the probability that $|\phi(q/p, T) - \tilde{\phi}(q/p)|$ is greater than δ does not exceed

$$\frac{1}{2pT\delta^2}\left\{1-\left(1-\frac{2k}{p}\right)^{2q}\right\},$$

which tends to 0 as $T \to \infty$ for any fixed δ.

For $S(u, T)$ we have

$$\operatorname{av}\{S(u,T)\}^2 = \frac{1}{\pi^2}\int\limits_0^T\int\limits_0^T \frac{\sin xu}{x}\frac{\sin yu}{y}\operatorname{av}\phi(x,T)\phi(y,T)\,dxdy,$$

and a similar result may be obtained.

In conclusion, we note the connexion between this theory and the work of G. I. Taylor* on the spectrum of turbulence. Taylor's R_x and $F(n)$ correspond to our $\bar{\phi}(x)$ and $\sqrt{(2\pi)}\bar{S}'(u)$ respectively. These are Fourier cosine transforms of each other, as is verified by Taylor's calculations. The fact that $F(n)$ is monotonic suggests that the appropriate analysis should be of the 'random function' type exemplified here, rather than of the direct type, which in the most natural cases leads to $S(u)$ being a step-function.

* G. I. Taylor, *Proc. Royal Soc.* A, 164 (1938), 476–90.